Robotics kinematics and Dynamics

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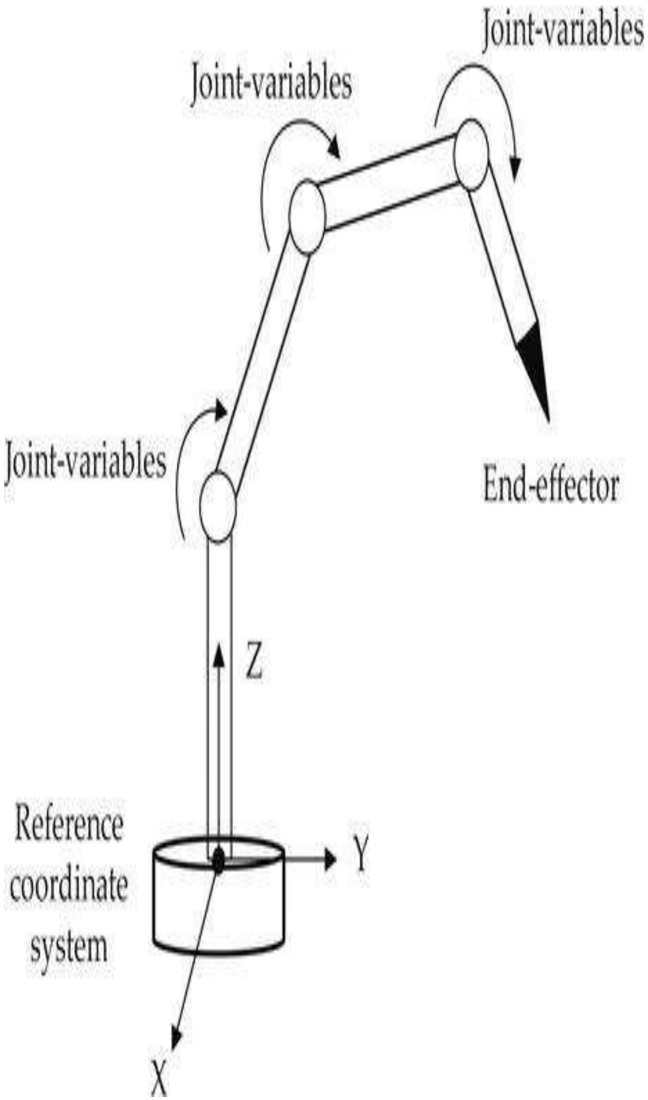
Robot kinematics

- KINEMATICS – the analytical study of the geometry of motion of a mechanism:
  - with respect to a fixed reference co-ordinate system,
  - without regard to the forces or moments that cause the motion.

- In order to control and programme a robot we must have knowledge of both its spatial arrangement and a means of reference to the environment.

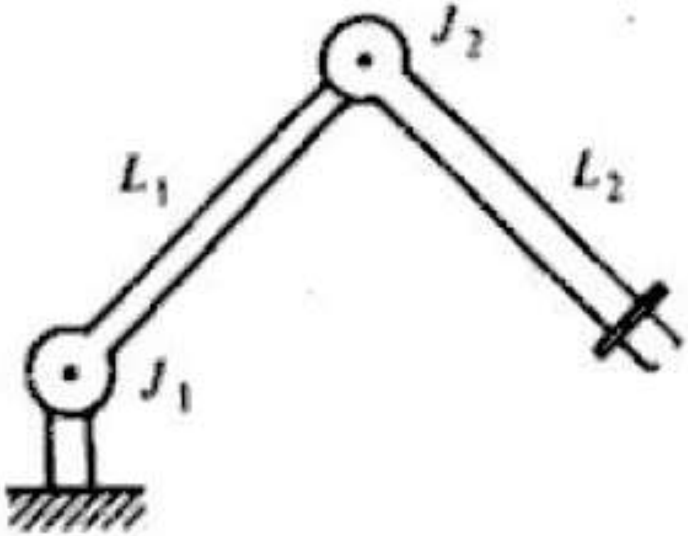
Open Chain Kinematics



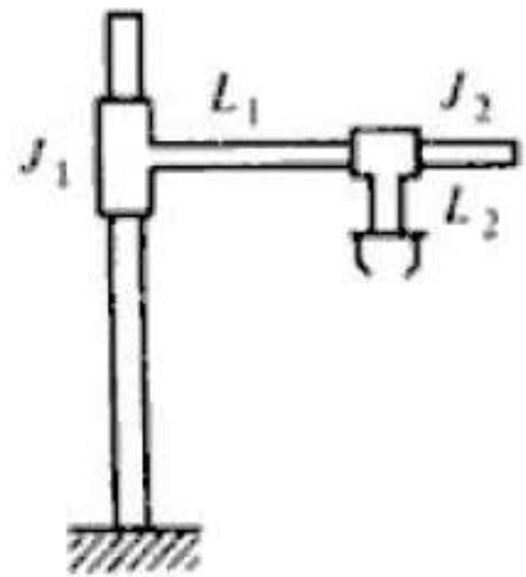
- Mechanics of a manipulator can be represented as a kinematic chain of rigid bodies (links) connected by revolute or prismatic joints.
- One end of the chain is constrained to a base, while an end effector is mounted to the other end of the chain.
- The resulting motion is obtained by composition of the elementary motions of each link with respect to the previous one

Robot kinematics

- Joint labeling: started from 1 and moving towards end effector, base being joint 1

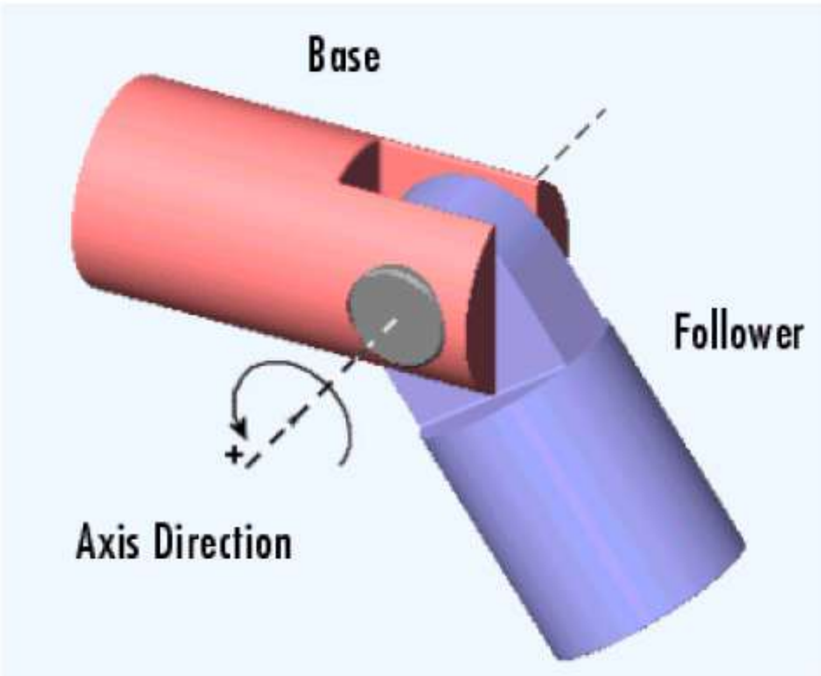


(a)

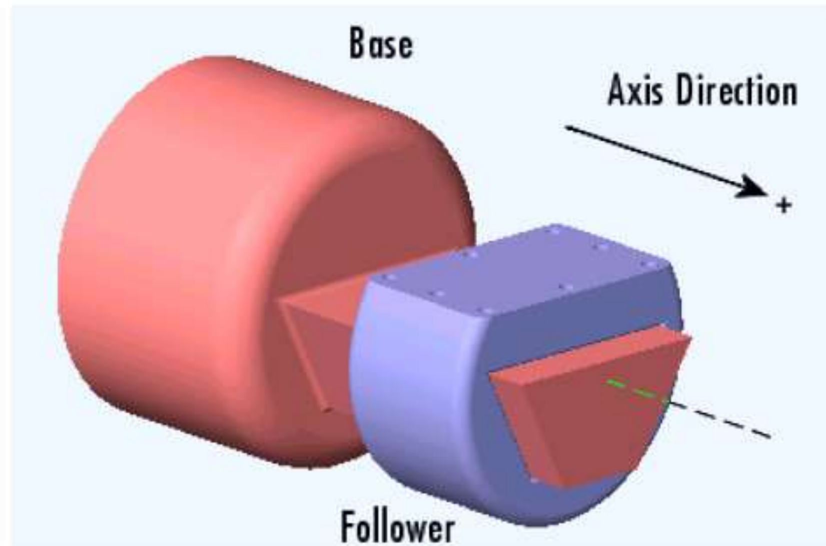


(b)

Two Basic Joints



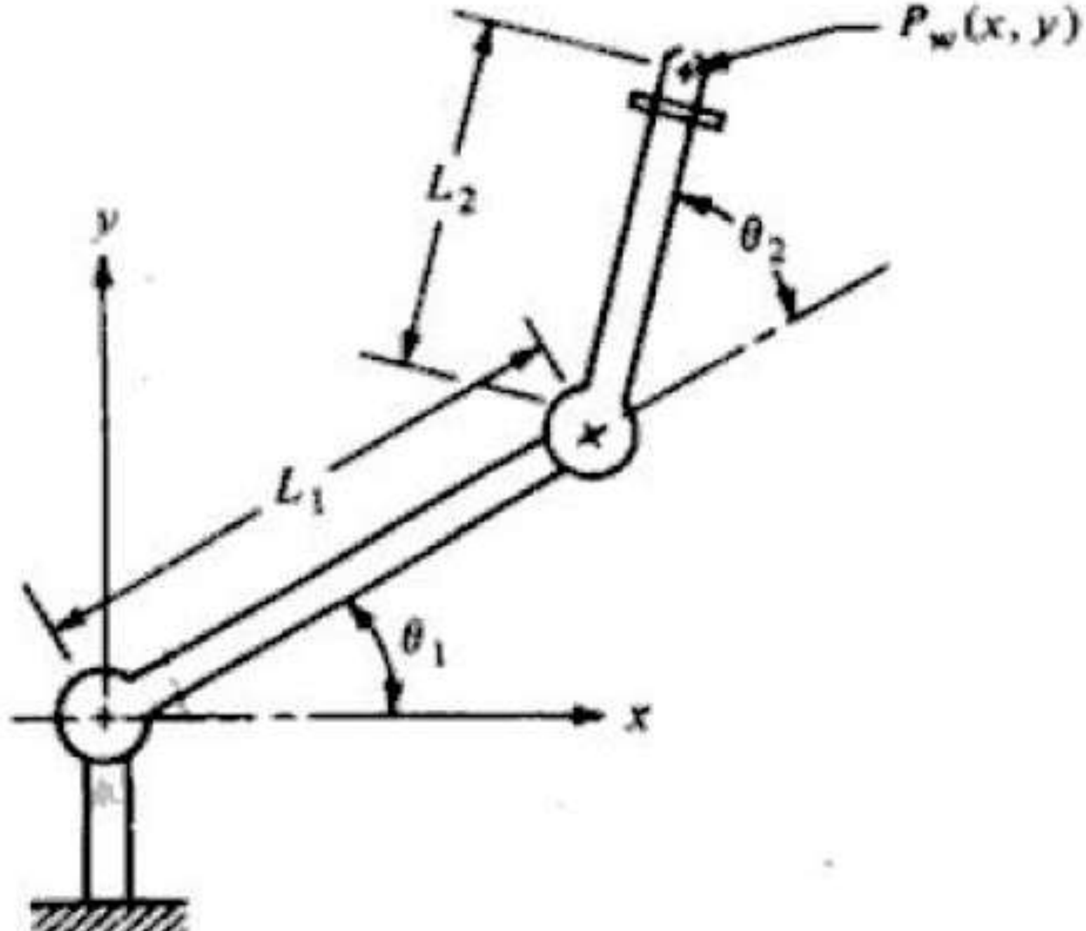
Revolute



Prismatic

Position representation

- Kinematics of RR robot is difficult compared to LL robot
- Analyzing in 2-D



- Position of end of the arm can be represented using:
- Joint space method: using joint angles

$$P_j = (\theta_1, \theta_2)$$

- World space : using cartesian coordinate system.

$$P_w = (x, y)$$

- Transformation from one representation to other is necessary for many application.

<u>Type of transformation:</u>

- Forward transformation or forward kinematics – going from joint space to world space
- Reverse transformation or inverse kinematics – going from world space to joint space.

- **Direct (also forward) kinematics** – Given are joint relations (rotations, translations) for the robot arm. Task: What is the orientation and position of the end effector?
- **Inverse kinematics** – Given is desired end effector position and orientation. Task: What are the joint rotations and orientations to achieve this?

Forward Transformation of a 2-Degree of Freedom Arm

We can determine the position of the end of the arm in world space by defining a vector for link 1 and another for link 2.

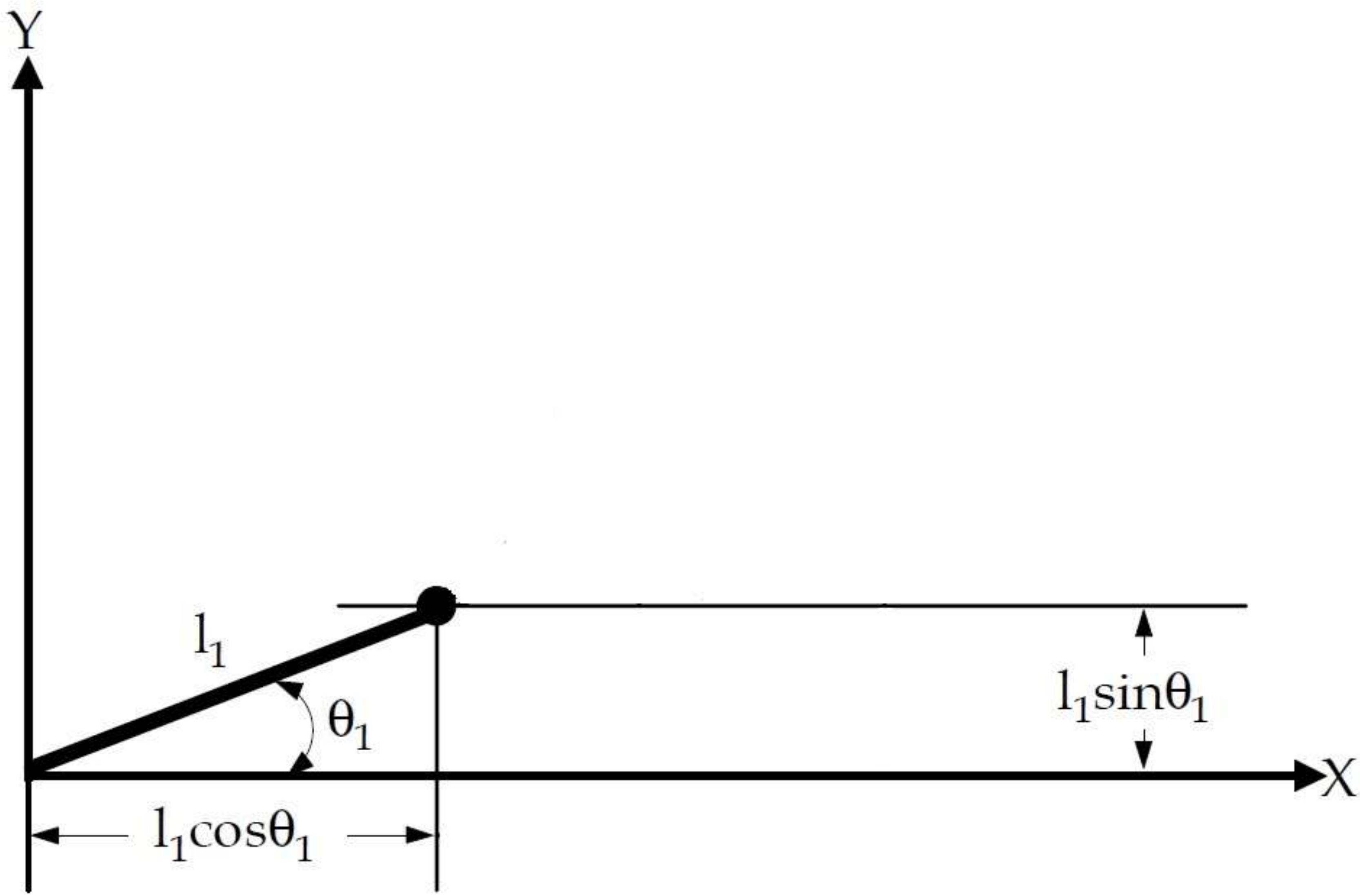
$$\tilde{\mathbf{r}}_1 = [L_1 \cos \theta_1, L_1 \sin \theta_1] \qquad (4\text{-}1)$$

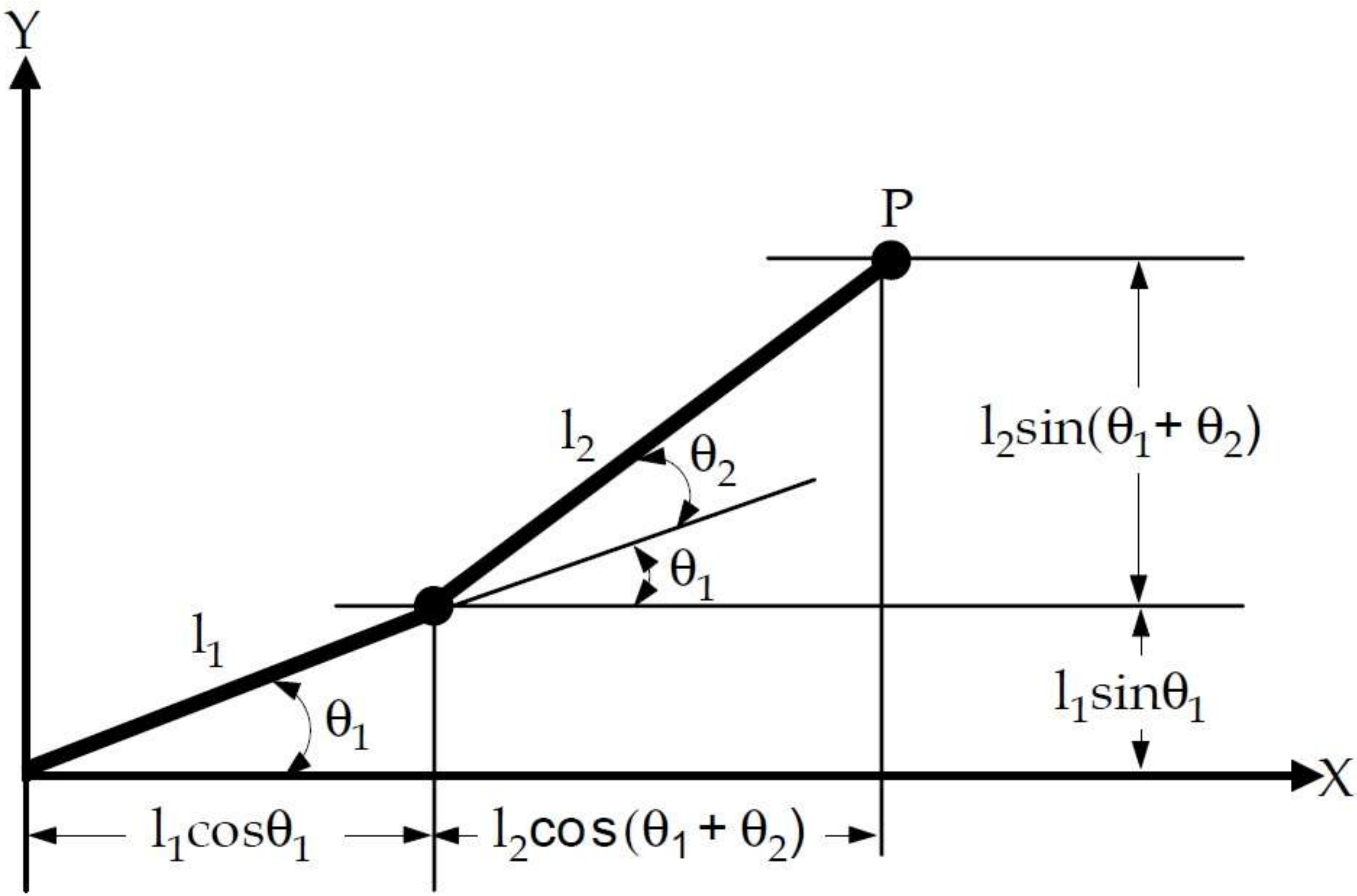
$$\mathbf{r}_2 = [L_2 \cos(\theta_1 + \theta_2), L_2 \sin(\theta_1 + \theta_2)] \qquad (4\text{-}2)$$

Vector addition of (4-1) and (4-2) yields the coordinates x and y of the end of the arm (point P_w) in world space

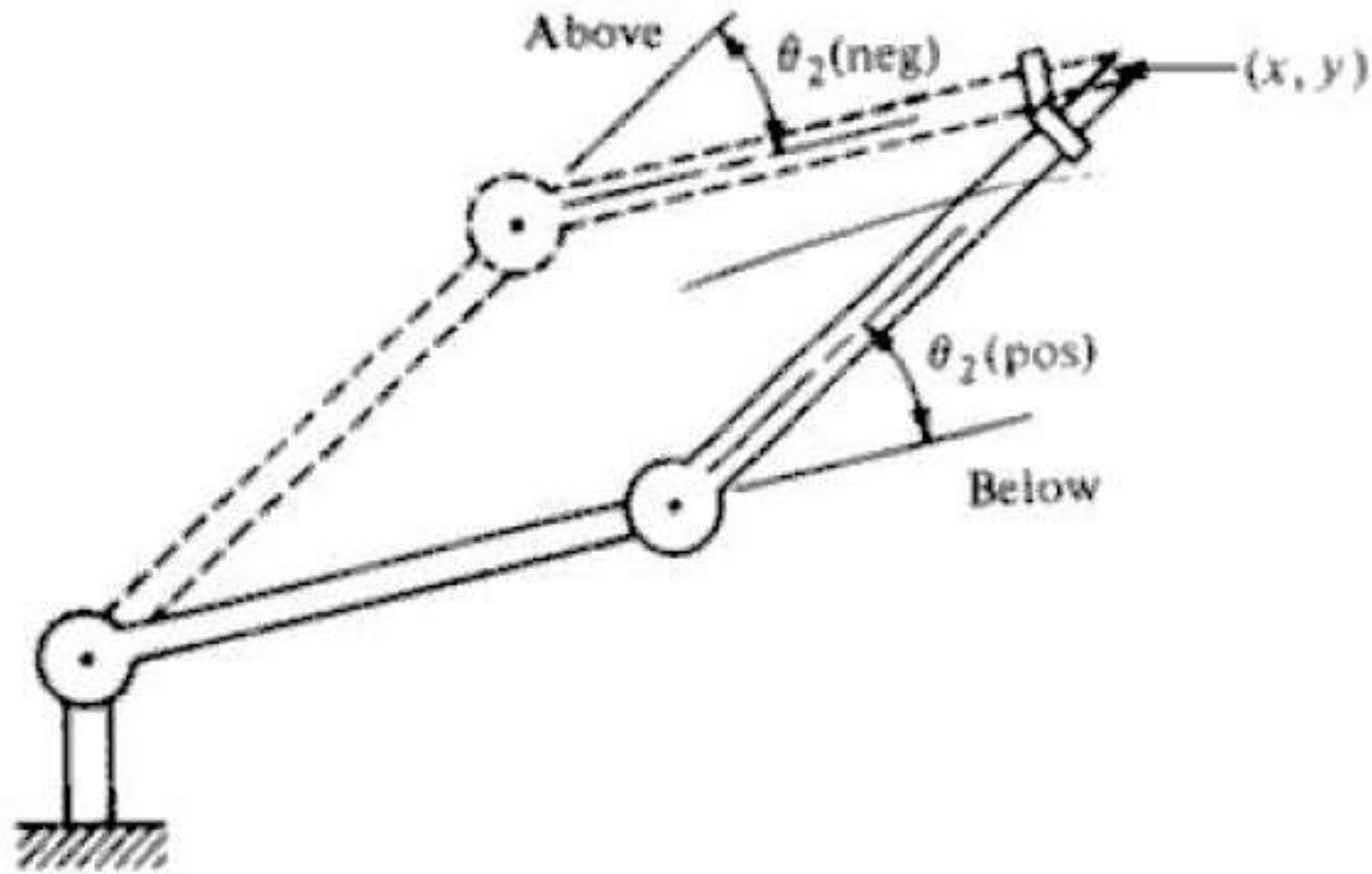
$$x = L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) \qquad (4\text{-}3)$$

$$y = L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2) \qquad (4\text{-}4)$$


Y
l_1
θ_1
$l_1\sin\theta_1$
X
$l_1\cos\theta_1$


Y
P
$l_2\sin(\theta_1+\theta_2)$
l_2
θ_2
θ_1
l_1
$l_1\sin\theta_1$
θ_1
X
$l_1\cos\theta_1$
$l_2\cos(\theta_1+\theta_2)$

Reverse transformation of 2 DOF arm



$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\sin(A + B) = \sin A \cos B + \sin B \cos A$$

we can rewrite Eqs. (4-3) and (4-4) as

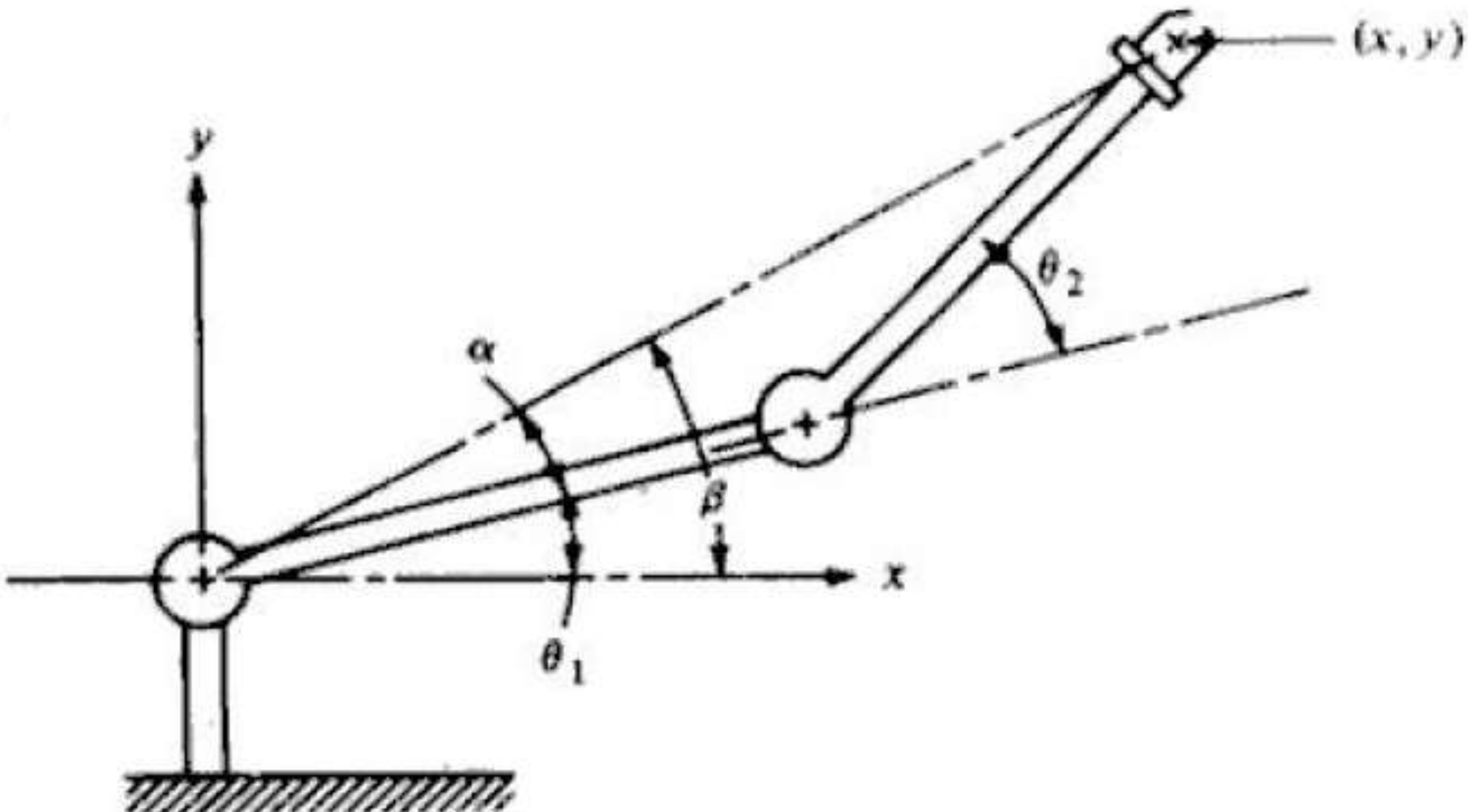
$$x = L_1 \cos \theta_1 + L_2 \cos \theta_1 \cos \theta_2 - L_2 \sin \theta_1 \sin \theta_2$$

$$y = L_1 \sin \theta_1 + L_2 \sin \theta_1 \cos \theta_2 + L_2 \cos \theta_1 \sin \theta_2$$

Squaring both sides and adding the two equations yields

$$\cos \theta_2 = \frac{x^2 + y^2 - L_1^2 - L_2^2}{2L_1L_2} \qquad (4\text{-}5)$$

Figure 4-4



Defining α and β as in Fig. 4-4 we get

$$\tan \alpha = \frac{L_2 \sin \theta_2}{L_2 \cos \theta_2 + L_1} \qquad (4\text{-}6)$$

$$\tan \beta = \frac{y}{x}$$

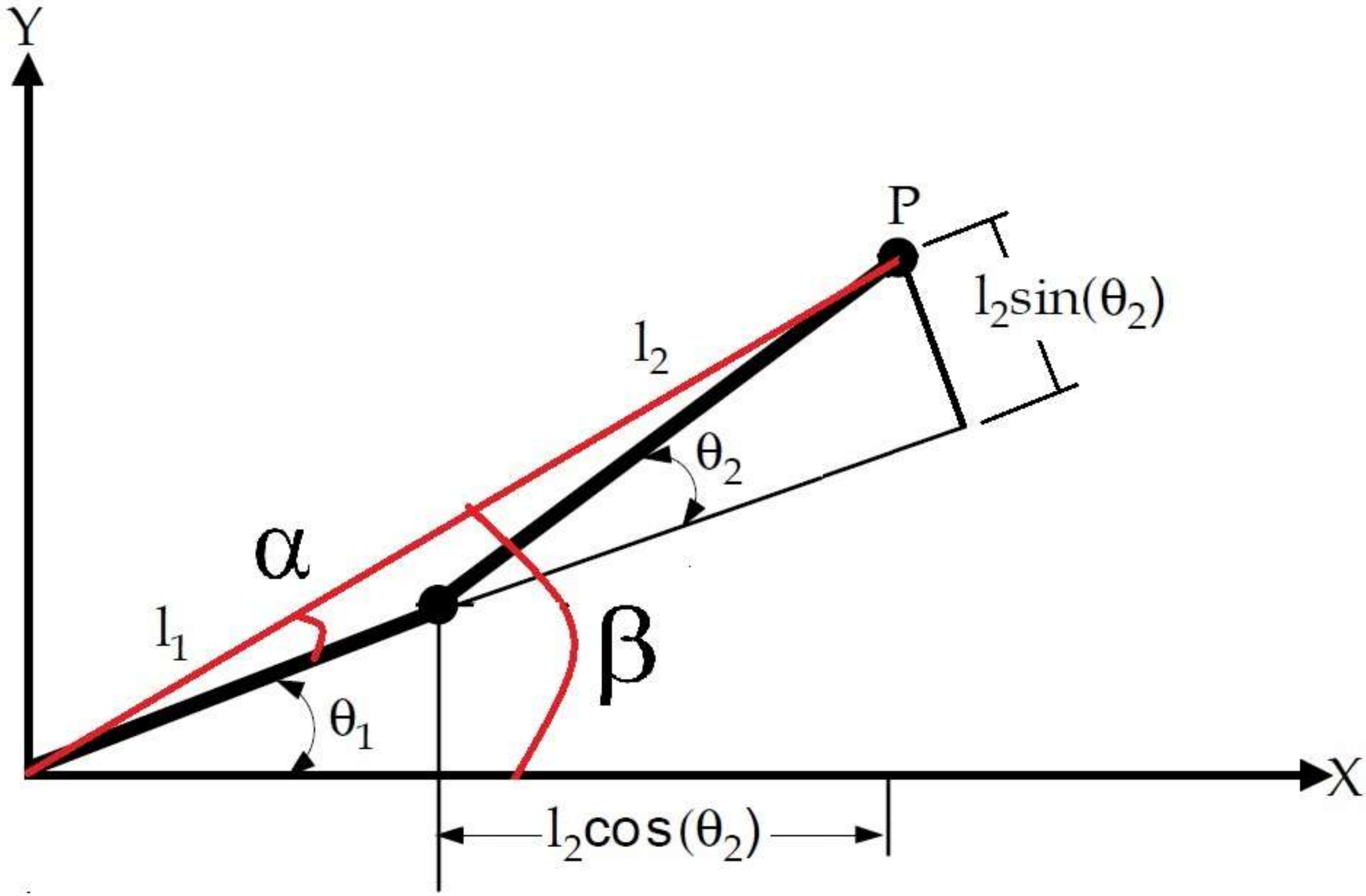
Using the trigonometric identity

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

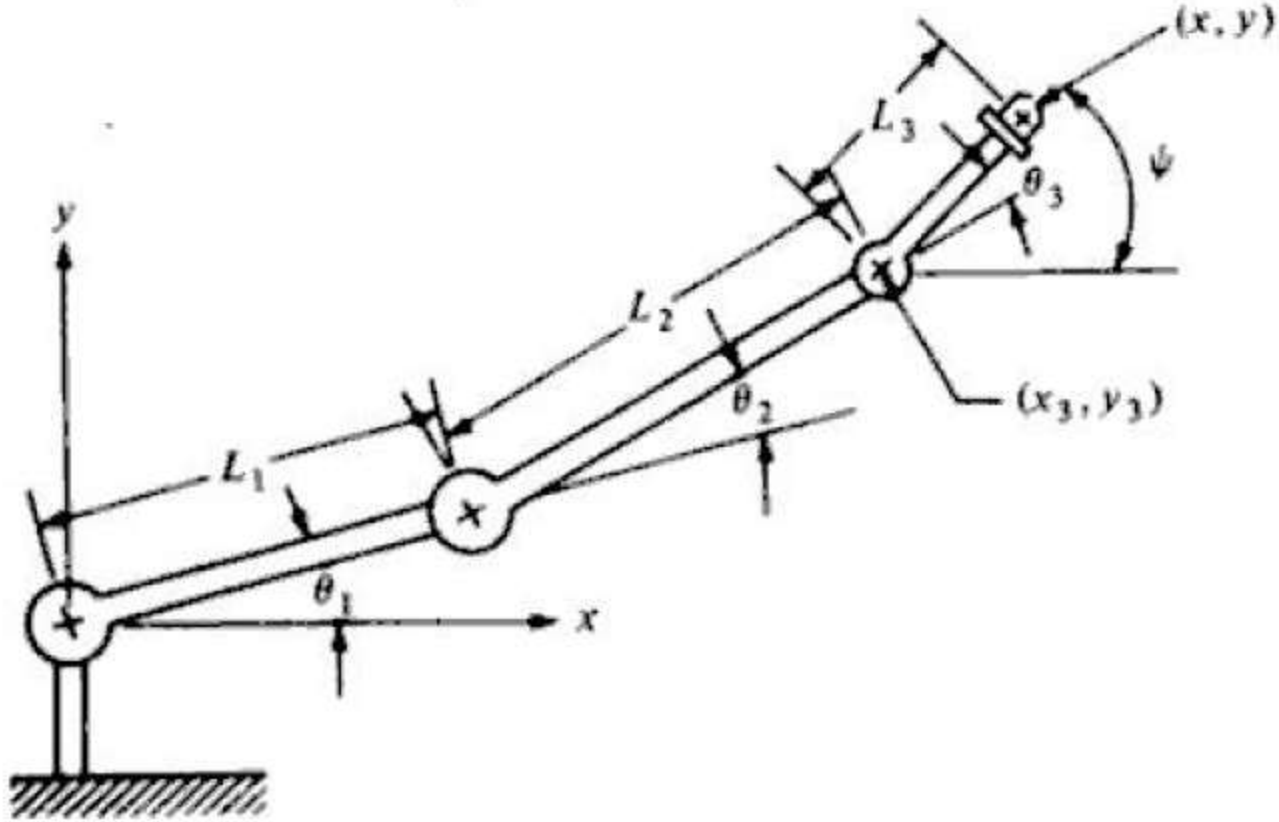
we get

$$\tan \theta_1 = \frac{[y(L_1 + L_2 \cos \theta_2) - xL_2 \sin \theta_2]}{[x(L_1 + L_2 \cos \theta_2) + yL_2 \sin \theta_2]} \qquad (4\text{-}7)$$

Knowing the link lengths L_1 and L_2 we are now able to calculate the required joint angles to place the arm at a position (x, y) in world space.


Y
P
$l_2\sin(\theta_2)$
l_2
θ_2
α
l_1
β
θ_1
X
$l_2\cos(\theta_2)$

3 DOF arm in two dimension



3 DOF arm in two dimension

Accordingly, we will incorporate a third degree of freedom into the previous configuration to develop the *RR:R* manipulator shown in Fig. 4-5. This third degree of freedom will represent a wrist joint. The world space coordinates for the wrist end would be

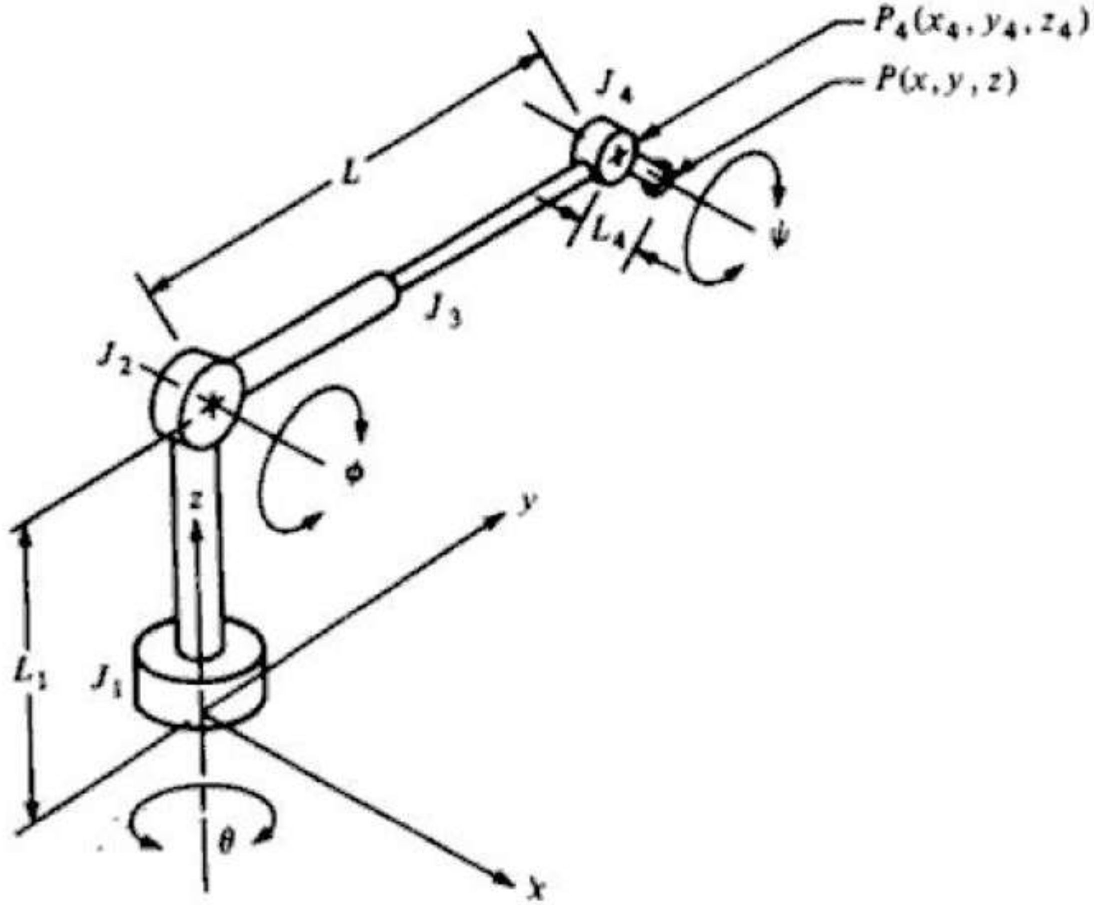
$$\left.\begin{aligned} x &= L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ y &= L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3) \\ \psi &= (\theta_1 + \theta_2 + \theta_3) \end{aligned}\right\} \quad (4\text{-}8)$$

We can use the results that we have already obtained for the 2-degree of freedom manipulator to do the reverse transformation for the 3-degree of freedom arm. When defining the position of the end of the arm we will use x, y, and ψ. The angle ψ is the orientation angle for the wrist. Given these three values, we can solve for the joint angles (θ_1, θ_2, and θ_3) using

$$x_3 = x - L_3 \cos \psi$$

$$y_3 = y - L_3 \sin \psi$$

4 DOF manipulator in three dimensions



The manipulator has 4 degrees of freedom: joint 1 (type T joint) allows rotation about the z axis; joint 2 (type R) allows rotation about an axis that is perpendicular to the z axis; joint 3 is a linear joint which is capable of sliding over a certain range; and joint 4 is a type R joint which allows rotation about an axis that is parallel to the joint 2 axis. Thus, we have a $TRL{:}R$ manipulator.

Let us define the angle of rotation of joint 1 to be the base rotation θ; the angle of rotation of joint 2 will be called the elevation angle ϕ; the length of linear joint 3 will be called the extension L (L represents a combination of links 2 and 3); and the angle that joint 4 makes with the $x-y$ plane will be called the pitch angle ψ. These features are shown in Fig. 4-6.

The position of the end of the wrist, P, defined in the world coordinate system for the robot, is given by

$$x = \cos\theta(L\cos\phi + L_4\cos\psi) \tag{4-9}$$

$$y = \sin\theta(L\cos\phi + L_4\cos\psi) \tag{4-10}$$

$$z = L_1 + L\sin\phi + L_4\sin\psi \tag{4-11}$$

Given the specification of point P (x, y, z) and pitch angle ψ, we can find any of the joint positions relative to the world coordinate system. Using P_4 (x_4, y_4, z_4), which is the position of joint 4, as an example,

$$x_4 = x - \cos\theta(L_4\cos\psi) \tag{4-12}$$

$$y_4 = y - \sin\theta(L_4\cos\psi) \tag{4-13}$$

$$z_4 = z - L_4\sin\psi \tag{4-14}$$

The values of L, ϕ, and θ can next be computed:

$$L = [x_4^2 + y_4^2 + (z_4 - L_1)^2]^{-1} \tag{4-15}$$

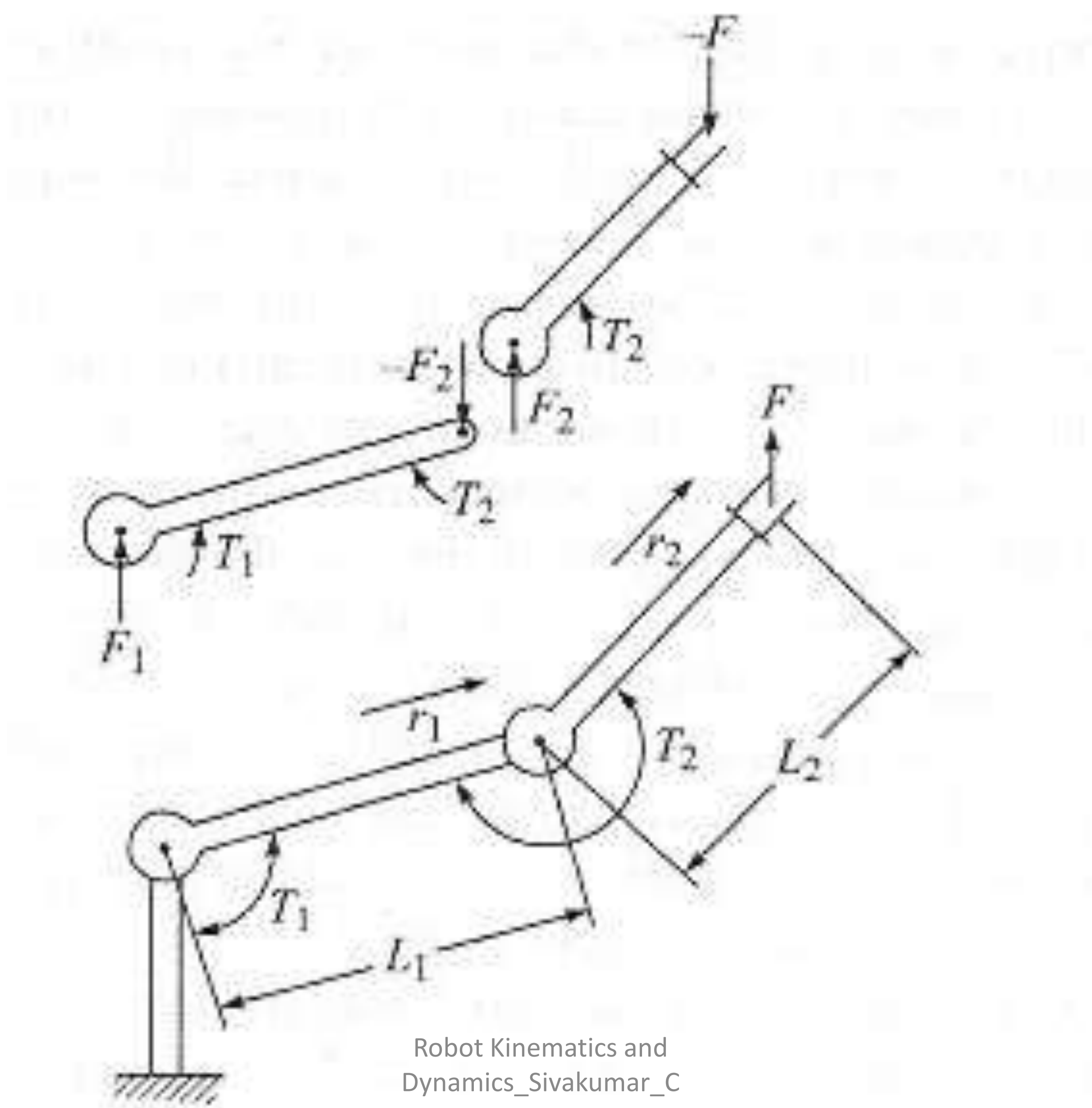
$$\sin\phi = \frac{z_4 - L_1}{L} \tag{4-16}$$

$$\cos\theta = \frac{y_4}{L} \tag{4-17}$$

Robot Dynamics

- Accurate control of manipulator depends on precise control of joints
- Control of joints depends on forces and intertias acting on them

a. Static analysis



Balancing the forces to know the torque

$$\mathbf{F}_1 - \mathbf{F}_2 = 0$$

$$\mathbf{F}_2 - \mathbf{F} = 0$$

$$\mathbf{F}_1 = \mathbf{F}_2 = \mathbf{F}$$

$$\mathbf{T}_1 = \mathbf{T}_2 + \mathbf{r}_1 \times \mathbf{F}$$

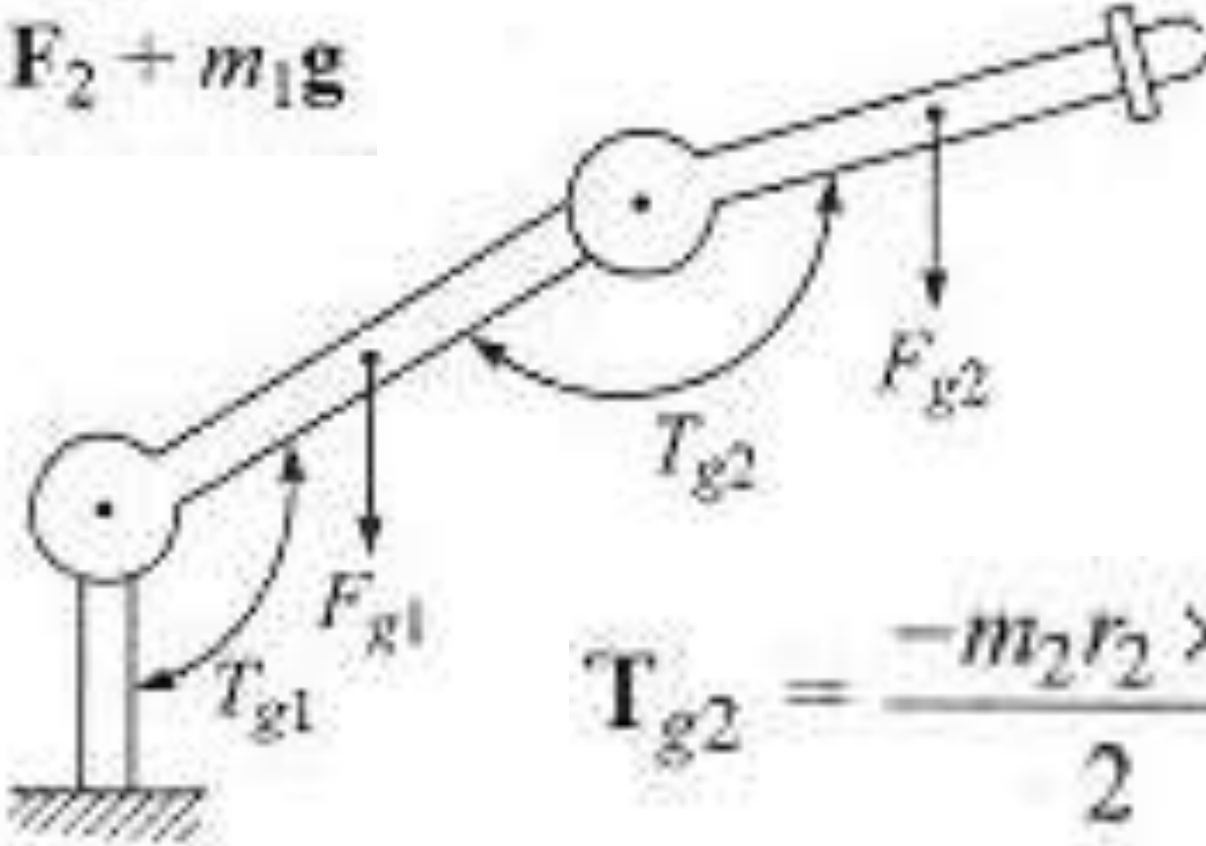
$$\mathbf{T}_2 = \mathbf{r}_2 \times \mathbf{F}$$

$$\mathbf{T}_1 = (\mathbf{r}_1 + \mathbf{r}_2) \times \mathbf{F}$$

Compensating for gravity

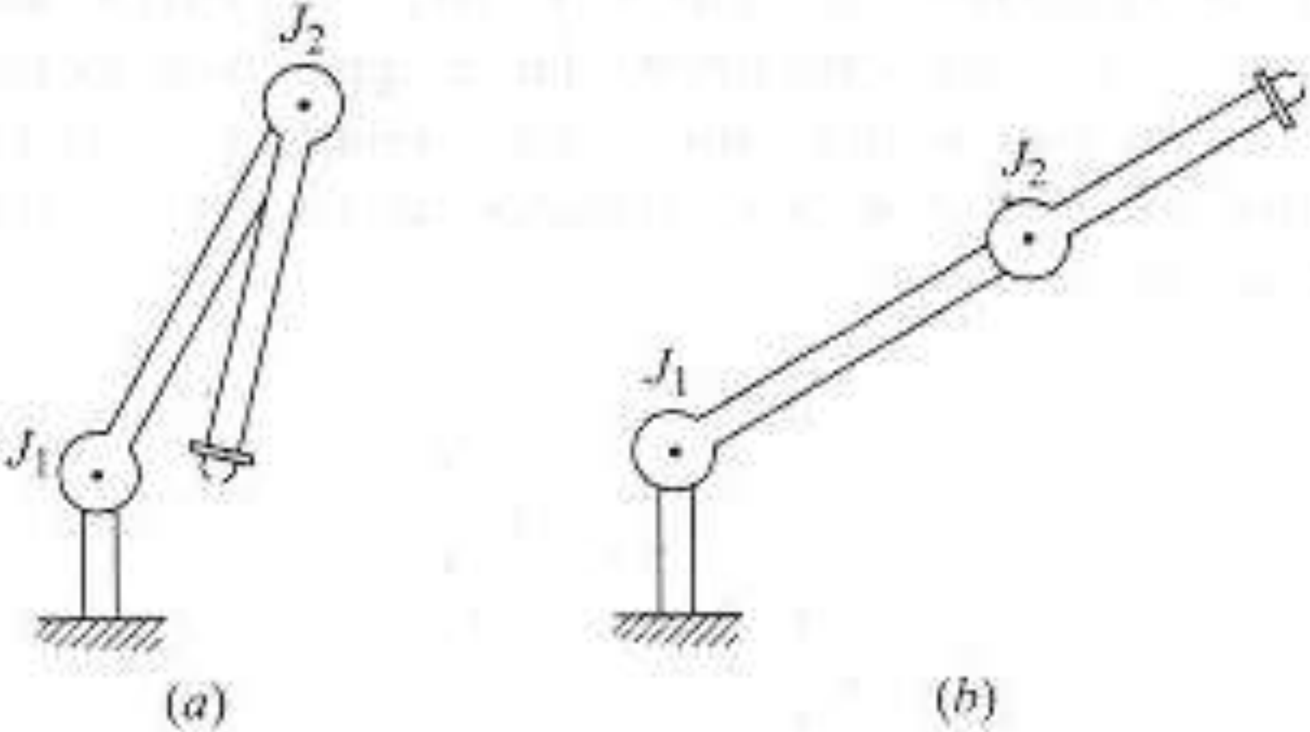
$$\mathbf{F}_2 = m_2\mathbf{g}$$

$$\mathbf{F}_1 = \mathbf{F}_2 + m_1\mathbf{g}$$



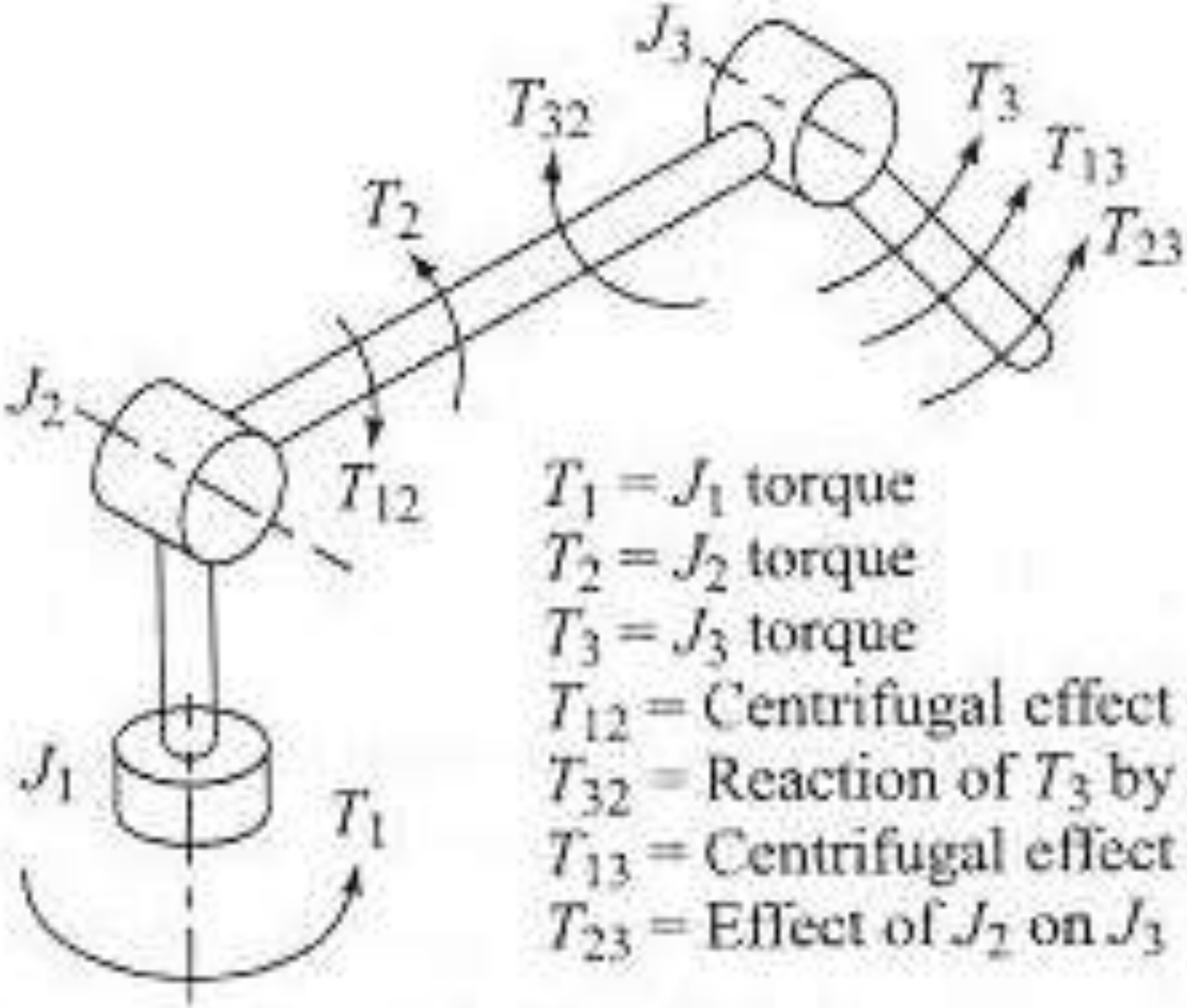
$$\mathbf{T}_{g2} = \frac{-m_2\mathbf{r}_2 \times \mathbf{g}}{2}$$

Robot arm dynamics



Arm inertias: (a) Minimum inertia about J_1, (b) Maximum inertia about J_1.

Torque requirement



$T_1 = J_1$ torque
$T_2 = J_2$ torque
$T_3 = J_3$ torque
T_{12} = Centrifugal effect of J_1 on J_2
T_{32} = Reaction of T_3 by J_2
T_{13} = Centrifugal effect of J_1 on J_3
T_{23} = Effect of J_2 on J_3

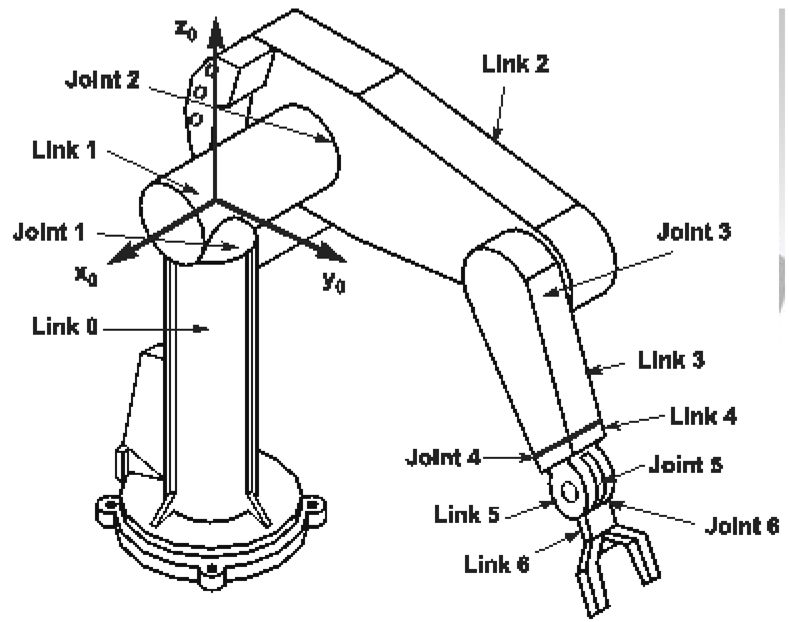
Kinematic

- Forward (direct) Kinematics
- Given: The values of the joint variables.
- Required: The position and the orientation of the end effector.

- Inverse Kinematics
- Given : The position and the orientation of the end effector.
- Required : The values of the joint variables.

Why DH notation

- Find the homogeneous transformation ***H*** relating the tool frame to the fixed base frame



Why DH notation

- A very simple way of modeling robot links and joints that can be used for any kind of robot configuration.

- This technique has became the standard way of representing robots and modeling their motions.

DH Techniques

1. Assign a reference frame to each joint (x-axis and z-axis). The D-H representation does not use the y-axis at all.
2. Each homogeneous transformation A_i is represented as a product of four basic transformations

DH Techniques

- Matrix A_i representing the four movements is found by: four movements

1. Rotation of θ about current Z axis
2. Translation of d along current Z axis
3. Translation of a along current X axis
4. Rotation of α about current X axis

$$A_i = Rot_{z,\theta_i} Trans_{z,d_i} Trans_{x,a_i} Rot_{x,\alpha_i}$$

$$R_{x,\theta} = Rot(x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix} \quad R_{z,\theta} = Rot(z,\theta) = \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

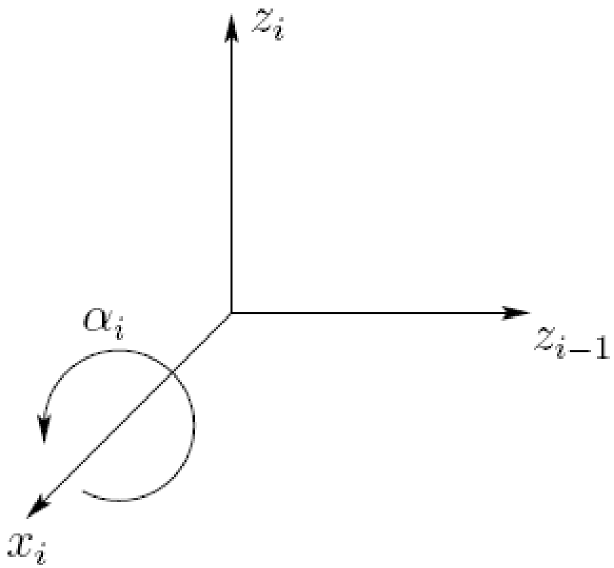
$$A_i = \begin{bmatrix} C\theta_i & -S\theta_i & 0 & 0 \\ S\theta_i & C\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha_i & -S\alpha_i & 0 \\ 0 & S\alpha_i & C\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

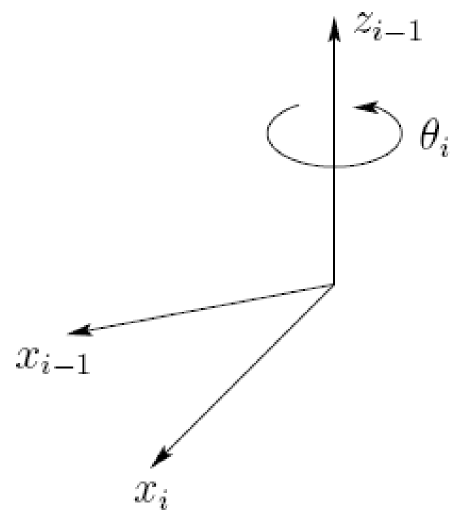
$$A_i = \begin{bmatrix} \mathrm{c}\theta_\mathrm{i} & \text{-}\mathrm{c}\alpha_\mathrm{i}\mathrm{s}\theta_\mathrm{i} & \mathrm{s}\alpha_\mathrm{i}\mathrm{s}\theta_\mathrm{i} & \mathrm{a}_\mathrm{i}\mathrm{c}\theta_\mathrm{i} \\ \mathrm{s}\theta_\mathrm{i} & \mathrm{c}\theta_\mathrm{i}\mathrm{c}\alpha_\mathrm{i} & \text{-}\mathrm{s}\alpha_\mathrm{i}\mathrm{c}\theta_\mathrm{i} & \mathrm{a}_\mathrm{i}\mathrm{s}\theta_\mathrm{i} \\ 0 & \mathrm{s}\alpha_\mathrm{i} & \mathrm{c}\alpha_\mathrm{i} & \mathrm{d}_\mathrm{i} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

DH Techniques

- The link and joint parameters :
- ***Link length*** $\mathbf{a}_i$ *: the offset distance between the* Z_{i-1} *and* Z_i *axes along the* X_i *axis.*
- ***Link offset*** $\mathbf{d}_i$ *the distance from the origin of frame* $i\text{–}1$ *to the* X_i *axis* along the Z_{i-1} *axis.*

DH Techniques





•***Link twist*** α_i *:the angle from the* Z_{i-1} *axis to the* Z_i *axis about the* X_i *axis.* The positive sense for α is determined from z_{i-1} and z_i by the right-hand rule.

•***Joint angle*** θ_i *the angle between the* X_{i-1} *and* X_i *axes about the* Z_{i-1} *axis.*

DH Techniques

- The four parameters:

a_i: link length, α_i: Link twist , d_i : Link offset and

θ_i : joint angle.

- The matrix A_i is a function of only a single variable q_i , it turns out that three of the above four quantities are constant for a given link, while the fourth parameter is the joint variable.

DH Techniques

- With the i[th] joint, a joint variable is q_i associated where

$$q_i = \begin{cases} \theta_i & : \text{joint i revolute} \\ d_i & : \text{joint i prismatic} \end{cases}$$

All joints are represented by the z-axis.

- If the joint is revolute, the z-axis is in the direction of rotation as followed by the right hand rule.
- If the joint is prismatic, the z-axis for the joint is along the direction of the liner movement.

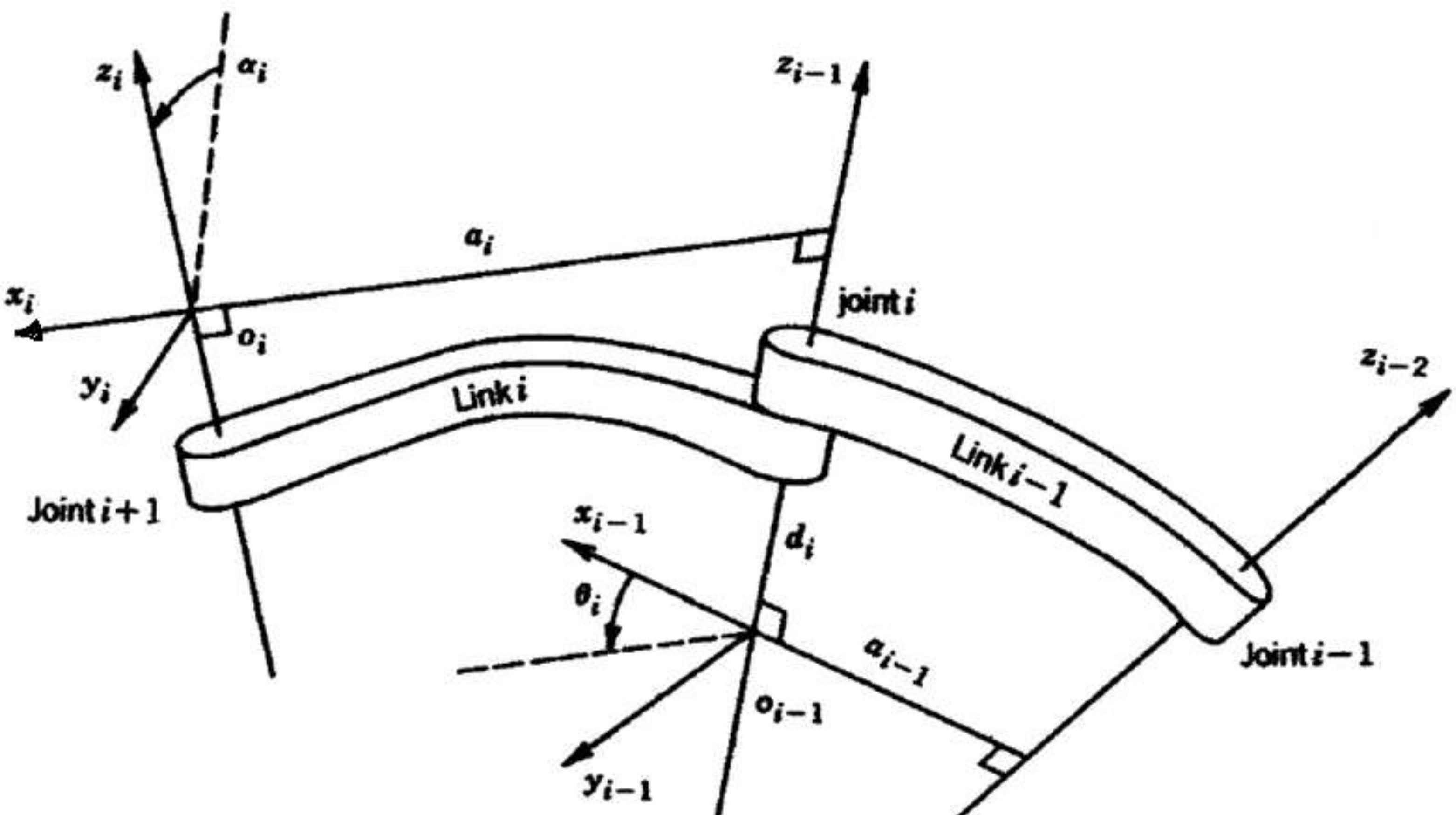
DH Techniques

3. Combine all transformations, from the first joint (base) to the next until we get to the last joint, to get the robot's *total transformation matrix*.

$$T_n^0 = A_1 . A_2 .... A_n$$

4. From T_n^0, the position and orientation of the tool frame are calculated.

DH Techniques



DH Techniques

Step 1: Locate and label the joint axes $z_0, \ldots, z_{n-1}$.

Step 2: Establish the base frame. Set the origin anywhere on the z_0-axis. The x_0 and y_0 axes are chosen conveniently to form a right-hand frame.

For $i = 1, \ldots, n-1$, perform Steps 3 to 5.

Step 3: Locate the origin o_i where the common normal to z_i and z_{i-1} intersects z_i. If z_i intersects z_{i-1} locate o_i at this intersection. If z_i and z_{i-1} are parallel, locate o_i in any convenient position along z_i.

Step 4: Establish x_i along the common normal between z_{i-1} and z_i through o_i, or in the direction normal to the $z_{i-1} - z_i$ plane if z_{i-1} and z_i intersect.

DH Techniques

Step 5: Establish y_i to complete a right-hand frame.

Step 6: Establish the end-effector frame $o_n x_n y_n z_n$. Assuming the n-th joint is revolute, set $z_n = \boldsymbol{a}$ along the direction z_{n-1}. Establish the origin o_n conveniently along z_n, preferably at the center of the gripper or at the tip of any tool that the manipulator may be carrying. Set $y_n = \boldsymbol{s}$ in the direction of the gripper closure and set $x_n = \boldsymbol{n}$ as $\boldsymbol{s} \times \boldsymbol{a}$. If the tool is not a simple gripper set x_n and y_n conveniently to form a right-hand frame.

Step 7: Create a table of link parameters a_i, d_i, α_i, θ_i.

$a_i =$ distance along x_i from o_i to the intersection of the x_i and z_{i-1} axes.

$d_i =$ distance along z_{i-1} from o_{i-1} to the intersection of the x_i and z_{i-1} axes. d_i is variable if joint i is prismatic.

$\alpha_i =$ the angle between z_{i-1} and z_i measured about x_i

DH Techniques

$\theta_i =$ the angle between x_{i-1} and x_i measured about z_{i-1} . θ_i is variable if joint i is revolute.

Step 8: Form the homogeneous transformation matrices A_i by substituting the above parameters into

$$A_i = \begin{bmatrix} c\theta_i & -c\alpha_i s\theta_i & s\alpha_i s\theta_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -s\alpha_i c\theta_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 9: Form $T_n^0 = A_1 \cdots A_n$. This then gives the position and orientation of the tool frame expressed in base coordinates.