CHAPTER 5
FLUID KINEMATICS & DYNAMICS

5.0 INTRODUCTION

- Kinematics is the geometry of motion.
- Kinematics of fluids is the branch of fluid mechanics that describes the motion of fluids and its consequences without consideration of the nature of forces causing the motion.
- A basic understanding of fluid kinematics forms the ground work for studies of the dynamic behavior of fluids in consideration of the forces accompanying the motion.

5.1 FLOW VISUALIZATION

- A flow field is a region in which the flow is defined at each and every point, at any instant of time.
- Usually velocity describes the flow
- In other words, a flow field is specified by velocities at different points in the region at different times.
- A fluid mass can be conceived of consisting of a number of fluid particles.
- Hence the instantaneous velocity at any point in a fluid region is actually the velocity of a particle that exists at that point at that instant of time.

- In order to ~~describe~~ obtain a complete picture of the flow, fluid motion is described by two methods: Lagrangian, and, Eulerian.

5.1.1 Lagrangian method

- In this method, the fluid motion is described by tracing the kinematic behavior of each and every individual particle constituting the flow.
- Identities of the particles are made by specifying their initial spatial position (location) at a given instant of time. The position of a particle at any other instant of time then becomes a function of its identity and time.
- The above statement can be analytically expressed as

$$\vec{s} = s(\vec{s}_0, t) \qquad (5.1)$$

where $\vec{s}$ is the position vector of a particle (with respect to a fixed point of reference) at a time t.

$\vec{s}_0$ is its initial position at a given time $t = t_0$, and thus specifies the identity of the particle.

- Equation (5.1) can be written in terms of scalar components (w.r.t. a rectangular cartesian coordinate system), as

$$x = x(x_0, y_0, z_0, t) \qquad (5.1a)$$

$$y = y(x_0, y_0, z_0, t) \qquad (5.1b)$$

$$z = z(x_0, y_0, z_0, t) \qquad (5.1c)$$

where x_0, y_0, z_0 are the initial coordinates and x, y, z are the coordinates at time t of the particle.

Hence $\vec{s}$ in (5.1) can be expressed as

$$\vec{s} = \hat{i}x + \hat{j}y + \hat{k}z$$

where $\hat{i}, \hat{j}, \hat{k}$ are the unit vectors along the x, y, z axes respectively.

- The velocity $\vec{v}$ and acceleration $\vec{a}$ of the fluid particle can be obtained from the material derivatives of the position of the particle w.r.t. time.

$$\vec{v} = \left[\frac{d\vec{s}}{dt}\right]_{s_0} \quad \text{------} (5.2)$$

$$\vec{a} = \left[\frac{d^2\vec{s}}{dt^2}\right]_{s_0} \quad \text{------} (5.3)$$

5.1.2 Eulerian method

- The Eulerian method is of greater advantage since it avoids the determination of the movement of each individual fluid particle. Instead it seeks the velocity $\vec{v}$ and its variation with time t at each and every location ($\vec{s}$) in the flow field.
- While in the Lagrangian view all hydrodynamic parameters are tied to the particles or elements, in the Eulerian view they are functions of location and time.

- Mathematically, the flow field in the Eulerian method is described as

$$\vec{V} = \vec{V}(\vec{S}, t) \quad \text{-----} \quad (5.4)$$

where $\vec{V} = \hat{i}u + \hat{j}v + \hat{k}w$

and $\vec{S} = \hat{i}x + \hat{j}y + \hat{k}w$ ---- (5.5)

Therefore,

$$\left.\begin{aligned} u &= u(x, y, z, t) \\ v &= v(x, y, z, t) \\ w &= w(x, y, z, t) \end{aligned}\right\} \quad \text{----} \quad (5.6)$$

- From eqns (5.4), (5.5) and (5.6), we can write

$$V(\vec{S}, t) = \frac{d\vec{S}}{dt} \quad \text{-----} \quad (5.7)$$

or,

$$\left.\begin{aligned} \frac{dx}{dt} &= u(x, y, z, t) \\ \frac{dy}{dt} &= v(x, y, z, t) \\ \frac{dz}{dt} &= w(x, y, z, t) \end{aligned}\right\} \quad \text{----} \quad (5.8)$$

- In principle, the Lagrangian method of description can be derived from the Eulerian method.

5.2 Lines of flow

5.2.1 Stream Lines

The Stream Line is defined as an imaginary line drawn in the flow field such that at a given ~~not~~ instant of time, the velocity vector is tangential to it. (See Fig. 5.1).

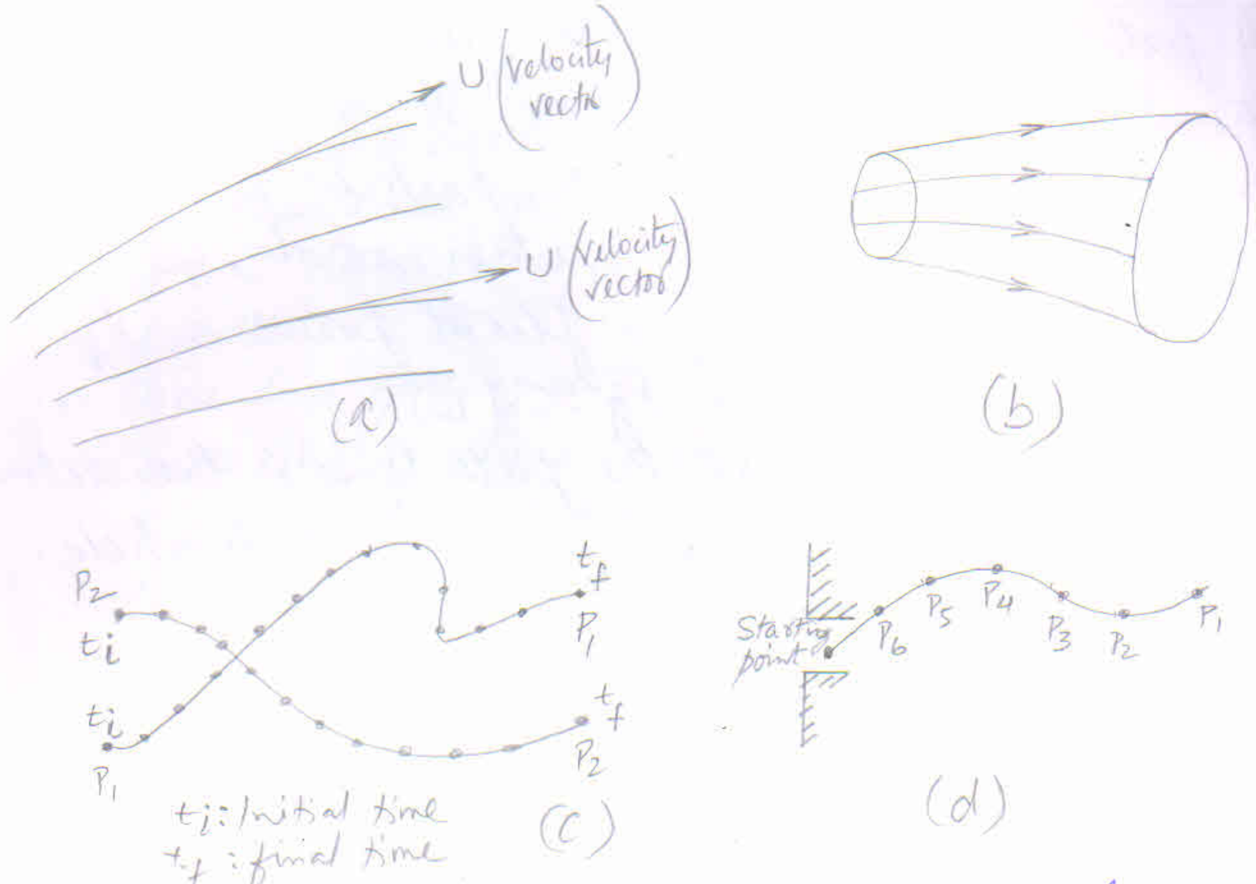

Fig. 5.1 (a) Streamlines in a flow field
(b) A stream tube
(c) Pathlines of points P_1 and P_2
(d) Streak lines

The streamline is therefore a line tangent to the velocity vector at every point in a given instant.

Stream tube

A series of streamlines which form a closed conduit, as shown in Fig. 5.1 (b).
Since the streamlines cannot be crossed by the flowing fluid, they act as a solid boundary within which the fluid flows. This is called a stream tube

5.2.2 Path lines

- A pathline is a line in the flow field describing the path of a fluid particle. It indicates the direction of the velocity of the particle at successive ~~int~~ instants of time. In other words, a path line is a trajectory of a fluid particle of fixed identity. See Fig. 5.1 (c).
- We can also say that the path line is the actual path traversed by a given fluid particle.

5.2.3 Streak lines

- A streak line is the locus of all particles in a flow field passing thru a given point at a given instant of time. Fig. 5.1(d).
- In other words, a streak line is a locus of all particles which had earlier passed thru a chosen point.

Note: While doing experiments in fluid mechanics, a dye is injected in the flow field to study the motion of fluid particles. In steady flow, all three lines are the same and there is no geometric distinction between the streamline, path line, and streak line.

Characteristics of stream lines and stream tubes

(a) Stream lines

- A streamline cannot intersect itself, nor can two streamlines cross
- There is no movement of the fluid mass across streamlines
- Streamline spacing varies inversely as the velocity. Converging of streamlines in any particular direction shows accelerated flow in that direction
- Whereas a pathline gives the path of one particular particle at successive instants of time, a streamline indicates the direction of a number of particles at the same instant.
- A series of streamlines represents the flow pattern at an instant

(b) Stream tube & stream surface

- A stream tube has finite dimensions
- As there is no flow perpendicular to streamlines, there is no flow across the surface (called stream ~~tube~~ surface) of the stream tube.
- The stream surface functions as if it were a solid wall.
- The shape of the stream tube changes from one instant to another because of the change in the position of streamlines.

Types of fluid flows

Fluid flows may be classified as follows:

1. Steady & unsteady flows
2. Uniform & non-uniform flows
   - The type of flow in which the velocity at any given time does not change w.r.t. position is called uniform flow.
3. One, two and three dimensional flows
4. Rotational and irrotational flows
   - A flow is said to be rotational if ~~the~~ fluid particles ~~which~~ while moving in the direction of flow rotate about their mass centers. Flow near solid boundaries is rotational.
5. Laminar and turbulent flows
   - A laminar flow is one in which the paths taken by individual particles do not cross one another and move along well-defined paths. This type of flow is also called streamline flow.
   - A turbulent flow is one in which fluid particles move such that their paths can cross one another.
   - Laminar and turbulent flows are characterized on the basis of Reynolds number $\left(Re = \frac{\rho V d}{\mu}\right)$

where $\rho \equiv$ density
$V \equiv$ velocity
$d \equiv$ characteristic length in the flow
$\mu \equiv$ viscosity

Laminar flow: $Re < 2000$
Turbulent " : $Re > 4000$
Transition " : $2000 < Re < 4000$

6. Compressible and incompressible flows

5.3 VELOCITY POTENTIAL

The velocity potential is defined as

$$-\phi = \int V_s \cdot ds \qquad (5.9)$$

in which V_s is the velocity along a small length element ds.

$$\therefore d\phi = -V_s \cdot ds$$

$$\text{or, } V_s = -\left(\frac{d\phi}{ds}\right)$$

The velocity potential is a scalar quantity dependent on space and time. Its negative derivative w.r.t. any direction gives the velocity in that direction.

$$u = -\frac{\partial \phi}{\partial x},\ v = -\frac{\partial \phi}{\partial y},\ w = -\frac{\partial \phi}{\partial z} \qquad (5.10)$$

The velocity potential ϕ thus provides an alternative means of expressing velocity components.

The minus sign in eqn. (5.10) appears because of the convention that the velocity potential decreases in the direction of flow just as the electric potential decreases in the direction in which the current flows. The velocity potential is not a physical quantity which could be directly measured, and therefore its zero position may be arbitrarily chosen.

The continuity equation for 3D steady incompressible flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{------} (5.11)$$

which may be written in terms of ϕ as

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0} \quad \text{---} (5.12)$$

Eqn. (5.12) is the 3D form of Laplace's eqn.

The angular velocities of a fluid element about its mass center are

$$\left.\begin{aligned} \omega_x &= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) ; \quad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \omega_z &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \end{aligned}\right\} (5.13)$$

The vorticity is defined as

$$\left.\begin{aligned} \xi_x &= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) ; \quad \xi_y = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \text{and} \quad \xi_z &= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \end{aligned}\right\} \quad \text{--- } 5.14)$$

The vorticity is therefore two times as great as the angular velocity.

If the vorticity or rate of rotation ω of each fluid particle about its mass center is zero, the flow must be irrotational. This sets up a condition for velocity potential to exist in a flow field.

The existence of velocity potential in a flow field ensures that the flow must be irrotational.

It is for this reason that an irrotational flow is often called potential flow.

Lines drawn in the flow field along which ϕ is constant are known as equipotential lines.

5.4 STREAM FUNCTION

The concept of stream function (ψ) is based on the principle of continuity and the properties of a streamline. It is applicable to 2D flow cases.

It can be shown that

$$d\psi = u\,dy - v\,dx \quad \text{------} \quad (5.15)$$

If ψ is considered as a function of x & y,

$$\psi = \psi(x, y),$$

$$\text{then } d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy \quad \text{---} \quad (5.16)$$

comparing (5.15) and (5.16)

$\partial\psi/\partial x$

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x} \quad \text{---} \quad (5.17)$$

Eqn. (5.17) states that the partial derivative of the stream function w.r.t. any direction gives the velocity component 90° clockwise w.r.t. that direction.

The mathematical expression for a streamline for 2D flow can be shown to be

$$\cancel{\frac{dx}{v} = \frac{dy}{u}}$$

$$\text{or, } \boxed{\frac{dy}{dx} = \frac{v}{u}} \quad \text{------} \quad (5.18)$$

Now, the continuity eqn. in two dimensions takes the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Substituting the values of u and v in terms of ψ, we get

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0 \quad \text{------} \quad (5.19)$$

Lines of constant ψ represent streamlines since there is no flow across these lines.

For irrotational flow:

$$\omega_z = 0$$

$$\therefore \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \quad \text{from eqn. (5.13)}$$

Expressing u and v in terms of ψ, we get

$$-\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \psi}{\partial y^2}$$

For irrotational flow

$$\text{or,} \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \text{------} \quad (5.20)$$

Eqn. (5.20) shows that the stream function must also satisfy the Laplace equation, provided that the flow is irrotational. Conversely, fluid flows which do not satisfy the Laplace eqn. in ψ are rotational flows.

5.5 CIRCULATION & VORTICITY

In rotational fluid motion, circulation is a very useful concept.

Circulation is defined as the line integral of the tangential component of the velocity taken around a closed contour. See Fig. 5.2.

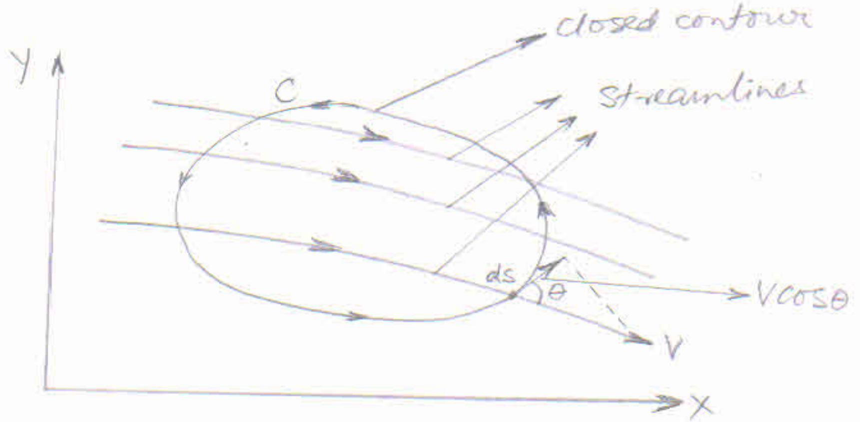

Fig. 5.2 Circulation concept

The limiting value of circulation, divided by the area of the closed contour, as the area tends to zero, is the vorticity along an axis normal to the area.

Circulation is taken as positive in the anti-clockwise direction.

Referring to Fig. 5.2,

$$\boxed{\Gamma = \oint_C \vec{V} \cdot d\vec{s} = \oint_C (u\,dx + v\,dy + w\,dz)} \quad \text{---} (5.21)$$

For 2D flow,

$$\Gamma = \oint_C V \cos\theta \, ds$$

$$= \oint_C (u\,dx + v\,dy) \quad \text{-----} (5.22)$$

where $\vec{V}$ = velocity in the flow field at the element ds

θ = angle between $\vec{V}$ and the tangent to the path (in the positive anti-clockwise direction along the path) at that point.

The vorticity is defined as

$$\boxed{\frac{\Gamma}{\text{Area of closed curve } C} = \text{Vorticity along the axis perpendicular to the plane containing the closed curve } C}$$

$\text{-----} (5.23)$

Relation between ψ and ϕ for 2D flow

- ϕ exists for irrotational flow only
- $u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$
- $v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$
- By continuity equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

- By irrotational flow condition

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

- ψ = constant along a streamline
- ϕ = constant along an equipotential line which is ~~normal~~ normal to streamlines.