CHAPTER 6. DIMENSIONAL ANALYSIS & SIMILITUDE

6.1 INTRODUCTION

- Solutions to engineering problems, due to their complex nature, are determined mostly from experiments.
- Due to economic considerations, it is not possible in a number of instances to perform experiments in the laboratory under identical conditions, in relation to operating parameters.
- Therefore, laboratory tests are usually carried out under altered conditions of the operating variables from the actual ones in practice.
- These variables in case of problems relating to fluid flow, are:
  - pressure
  - velocity
  - geometrical dimensions of the working system
  - physical properties of the working fluid
- Questions that arise out of this situation are:
  - How can we apply test results from laboratory experiments to the actual problem (which has a different set of conditions)?
  - When the performance of a system is governed by a large number of operating parameters such as input variables, a large number of

experiments are required to determine the influences of each and every operating parameter on the performance of the system. Is it possible, by any means, to reduce the large number of experiments, to a lesser number, in achieving the same objective?

6.2 PHYSICAL SIMILARITY

- The answer to the above questions lies in the principle of physical similarity.
- This principle makes it possible and justifiable to (a) apply results taken from tests under one set of conditions, to another set of conditions, and
  (b) to predict the influences of a large number of independent operating variables on the performance of a system, from an experiment with a limited number of operating variables.
- A large part of the progress made in fluid mechanics and engineering applications has come from experiments conducted on scale models.
- NO aircraft is now built before exhaustive tests are carried out on small models in a Wind Tunnel.
- The behavior and power requirements of a ship are calculated in advance from results of tests in which a small model of the ship is towed thru water, etc.

- In a number of situations, tests are conducted with one fluid, and the results are applied to situations in which another fluid is used.

6.3 CONCEPT & TYPES OF PHYSICAL SIMILARITY

- The primary and fundamental requirement for physical similarity between two problems is that the physics of the problems must be the same.
- For example, a fully developed flow thru a closed conduit can never be considered to be, under any situation, physically similar with a flow in an open channel. Reason: the flow in the closed conduit is governed by viscous and pressure forces, while for the open channel gravity force is dominant to maintain the flow.
- Therefore, the laws of similarity have to be sought between problems described by the same physics.
- General proposition: Two systems, described by the same physics, but operating under different sets of conditions, are said to be physically similar w.r.t. certain specified physical quantities, when the ratio of corresponding magnitudes of these quantities between the two systems is the same everywhere.

- If the specified physical quantities are geometric dimensions, the similarity is called geometric similarity.

  If the quantities are related to motions, the similarity is called kinematic similarity.

  If the quantities refer to forces, the similarity is called dynamic similarity.
- In the field of mechanics, these three similarities together constitute the complete similarity between problems of the same kind.

6.3.1 GEOMETRIC SIMILARITY

- Geometric similarity is the similarity of shape.
- In geometrically similar systems, the ratio of any length in one system to the corresponding length in any other system, is the same everywhere. This ratio is called the scale factor.
- Geometrically similar objects are similar in shapes, i.e. they are proportionate in their physical dimensions, but differ in size.
- In investigations of physical similarity, the full size or actual size systems are called prototypes, while the laboratory scale systems are called models.
- Use of the same fluid with both the prototype are not and the model is not necessary, nor is the model necessarily smaller than the prototype. Example : flow through an

injection nozzle or a carburettor. This would be more easily studied using a model much larger than the prototype.

- The model and prototype may be of identical size, although the two may then differ in regard to other factors such as velocity, and properties of the fluid.
- If l_1 and l_2 are the two characteristic physical dimensions of any object, then the requirement of geometric similarity is (Fig. 6.1)

$$\frac{l_{1_m}}{l_{1_p}} = \frac{l_{2_m}}{l_{2_p}} = l_r = \text{Scale factor or model ratio}$$

Where m denotes the model
& p " " prototype

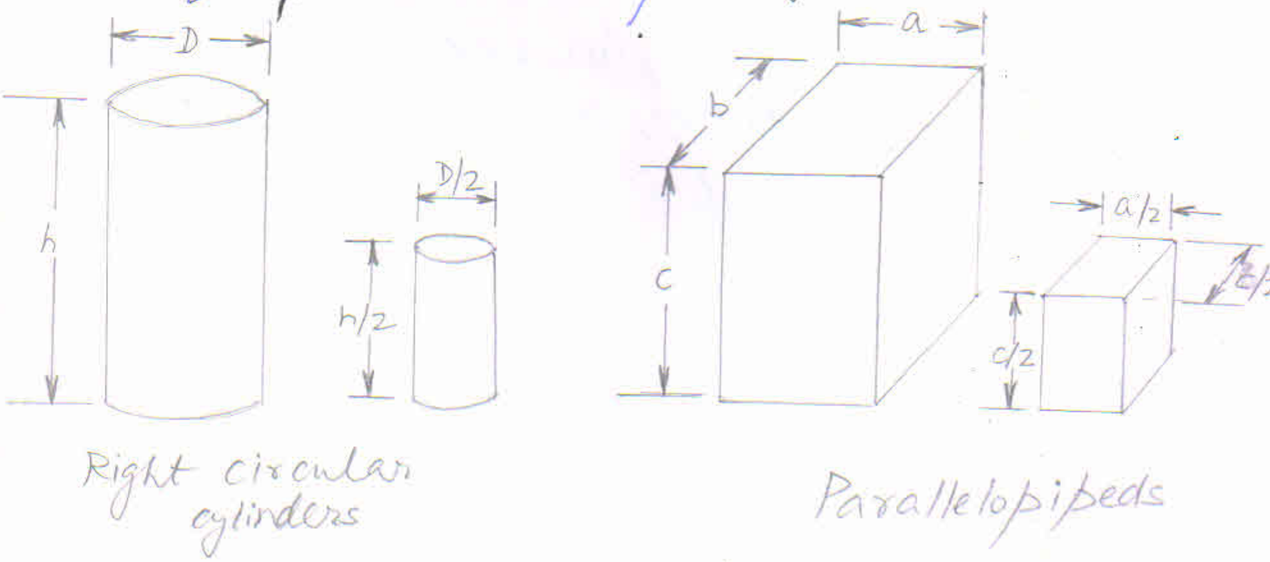

Fig 6.1 Geometrically similar Objects

- The roughness of the surface should also be geometrically similar, though it may difficult to ~~achieve~~ achieve this in all cases.. If for any reason the scale factor is not the same throughout, a distorted model results.
- Sometimes it may happen that to have perfect geometric similarity within the available laboratory space, the physics of the problem changes. For example, in case of large prototypes, such as rivers, the size of the model is limited by the available floor space of the laboratory. But if a very low scale factor is used in reducing both horizontal and vertical lengths, this may result in a stream so shallow that surface tension has considerable effect, and in addition, the flow may be laminar in the model, instead of turbulent in the real river.
- The extent to which perfect geometric similarity is to be sought therefore depends on the problem being investigated, and the accuracy required from the solution.

6.3.2 KINEMATIC SIMILARITY

- Kinematic similarity is similarity of motion.
- Since motion is described by distance and time, kinematic similarity implies similarity of lengths (i.e. geometric similarity), and, in addition, similarity of time intervals.
- If the corresponding lengths in two systems are in a fixed ratio, the velocities of corresponding particles must be in a fixed ratio of magnitude of corresponding time intervals.
- If the ratio of corresponding lengths, known as the scale factor, is l_r, and the ratio of corresponding time intervals is t_r, then the magnitudes of corresponding velocities are in the ratio of l_r/t_r, and the magnitudes of corresponding accelerations are in the ratio of l_r/t_r^2.
- Example of kinematic similarity : Planetarium
- When fluid motions are kinematically similar, the patterns formed by streamlines are geometrically similar at corresponding times.

6.3.3 DYNAMIC SIMILARITY

- Dynamic similarity is the similarity of forces.
- In dynamically similar systems, the magnitudes of forces at similar points in each system are in a fixed ratio.
- In other words, the ratio of magnitudes of any two forces in one system, must be the same as the ratio of the magnitude of the corresponding forces in the other system.
- In a system involving flow of a fluid, different forces due to different causes may act on a fluid element. These forces are as follows:

  - Viscous force $\vec{F}_v$
  - Pressure force $\vec{F}_p$
  - Gravity force $\vec{F}_g$
  - Capillary force $\vec{F}_c$ (due to surface tension)
  - Compressibility force $\vec{F}_e$ (due to elasticity)

- According to Newton's law, the resultant $\vec{F}_R$ of all these forces will cause acceleration of a fluid element.

  Hence $\vec{F}_R = \vec{F}_v + \vec{F}_p + \vec{F}_g + \vec{F}_c + \vec{F}_e$ --- (6.1)

- Moreover, the inertia force $\vec{F}_i$ is defined as equal and opposite to the resultant accelerating force $\vec{F}_R$. Therefore, eqn (6.1) may be expressed as

$$\vec{F}_R = -\vec{F}_i$$

or, $$\vec{F}_v + \vec{F}_p + \vec{F}_g + \vec{F}_c + \vec{F}_e + \vec{F}_i = 0 \quad \text{--- (6.1a)}$$

- For dynamic similarity, the magnitude ratios of these forces have to be the same for both prototype and model. The inertia force $\vec{F}_i$ is usually taken as the common denominator to describe the ratios as

$$\frac{|\vec{F}_v|}{|\vec{F}_i|};\ \frac{|\vec{F}_p|}{|\vec{F}_i|};\ \frac{|\vec{F}_g|}{|\vec{F}_i|};\ \frac{|\vec{F}_c|}{|\vec{F}_i|};\ \frac{|\vec{F}_e|}{|\vec{F}_i|}$$

- Fluid motion, under all such forces, is characterized by
  - (a) hydrodynamic parameters like pressure, velocity, and acceleration due to gravity
  - (b) rheological and other physical properties of the fluid involved
  - (c) geometrical dimensions of the system

study of the flow of matter, primarily in a liquid state, but also as "soft solids."

- The next step is to express the magnitudes of different forces in terms of these parameters, so as to know the extent of their influences on different forces acting on a fluid element in the course of its flow.

6.4 DIMENSIONLESS NUMBERS

The forces mentioned in the previous section can be represented by the following dimensional equations:

- Viscous force $\vec{F_v}$

$|\vec{F_v}|$ = shear stress × surface area over which the shear stress acts

= (viscosity × rate of shear strain) × surface area over which shear stress acts

= (μ × velocity gradient) × surface area ...

$\propto \mu \times \frac{V}{l} \times l^2$

i.e. $|\vec{F_v}| \propto \mu V l$ --- (6.2a) (where V is velocity, & l is a characteristic length in the flow)

- Pressure force $\vec{F_p}$

$|\vec{F_p}| \propto \Delta p \cdot l^2$ --- (6.2b) (where Δp is some characteristic pressure difference in the flow)

- Gravity force $\vec{F_g}$

$|\vec{F_g}| \propto \rho l^3 g$ --- (6.2c)

- Capillary or surface tension force $\vec{F_c}$

$|\vec{F_c}| \propto \sigma l$ --- (6.2d)

surface tension $\left[\frac{N}{m}\right]$

- <u>Compressibility or elastic force $\vec{F_e}$</u>

Elastic force comes into consideration due to the compressibility of the fluid in course of its flow. For a given compression, the increase in pressure is proportional to the bulk modulus of elasticity E, and gives rise to a force known as elastic force.

$$|\vec{F_e}| \propto E l^2 \quad \text{- - - - - - - - - (6.2e)}$$

- <u>Inertia force $\vec{F_i}$</u>

The inertia force acting on a fluid element is equal in magnitude to the mass of the element multiplied by its acceleration.

Now, mass x acceleration

$$= \rho l^3 \times \frac{V}{t} = \rho l^3 \times V \times \frac{1}{l/V} \quad \left(\because V = \frac{l}{t}\right)$$

(where t=time)

$$= \rho l^2 V^2$$

$$\therefore |\vec{F_i}| \propto \rho l^2 V^2 \quad \text{- - - - - - - - - (6.2f)}$$

Now, the flow of a fluid in practice does not involve all the forces listed simultaneously. Therefore, the pertinent dimensionless parameters for dynamicsemilarity are derived from the ratios of dominant forces causing or governing the flow.

6.4.1 Dynamic similarity of flows governed by viscous, pressure and inertia forces

$$\frac{\text{Viscous force}}{\text{Inertia force}} = \frac{|\vec{F_v}|}{|\vec{F_i}|} \propto \frac{\mu V l}{\rho V^2 l^2} = \frac{\mu}{\rho l V} \quad \text{---} (6.3a)$$

$$\frac{\text{Pressure force}}{\text{Inertia force}} = \frac{|\vec{F_p}|}{|\vec{F_i}|} = \frac{\Delta p\, l^2}{\rho l^2 V^2} = \frac{\Delta p}{\rho V^2} \quad \text{---} (6.3b)$$

$$\boxed{\text{Reynolds number (Re)} = \frac{\rho l V}{\mu} = \frac{\text{Inertia force}}{\text{Viscous force}}} \quad \text{---} (A)$$

$$\boxed{\text{Euler number (Eu)} = \frac{\Delta p}{\rho V^2} = \frac{\text{Pressure force}}{\text{Inertia force}}} \quad \text{---} (B)$$

Thus, for complete dynamic similarity to exist between the prototype and the model for this class of flows, the Reynolds number and the Euler number have to be the same for the prototype and the model.

Thus $$\frac{\rho_p l_p V_p}{\mu_p} = \frac{\rho_m l_m V_m}{\mu_m}$$

& $$\frac{\Delta p_p}{\rho_p V_p^2} = \frac{\Delta p_m}{\rho_m V_m^2}$$

6.4.2 Dynamic similarity of flows with Gravity, Pressure, and Inertia forces

$$\frac{\text{Gravity force}}{\text{Inertia force}} = \frac{|\vec{F_g}|}{|\vec{F_i}|} \propto \frac{\rho l^3 g}{\rho l^2 V^2} = \frac{lg}{V^2} \quad \text{--(6.3c)}$$

$$\boxed{\text{Froude number } (Fr) = \frac{V}{\sqrt{lg}}} \quad \text{------ (C)}$$

6.4.3 Dynamic similarity of flows with Surface tension as the dominant force

$$\frac{\text{Surface tension force}}{\text{Inertia force}} = \frac{|\vec{F_c}|}{|\vec{F_i}|} = \frac{\sigma l}{\rho l^2 V^2} = \frac{\sigma}{\rho V^2 l} \quad \text{---(6.3d)}$$

$$\boxed{\text{Weber number } (Wb) = \frac{\sigma}{\rho V^2 l}} \quad \text{------ (D)}$$

6.4.4 Dynamic similarity of flows with elastic force

$$\frac{\text{Elastic force}}{\text{Inertia force}} = \frac{|\vec{F_e}|}{|\vec{F_i}|} \propto \frac{E l^2}{\rho l^2 V^2} = \frac{E}{\rho V^2} \quad \text{----6.3e)}$$

$$\boxed{\text{Cauchy number} = \frac{\rho V^2}{E}} \quad \text{------ (E)}$$

6.5 METHODS OF DIMENSIONAL ANALYSIS

- In the dimensional analysis of a physical phenomenon, the relationship between the dependent and independent variables is studied in terms of their basic dimensions to obtain information about the functional relationship between the dimensionless parameters that control the phenomenon.
- There are several methods of reducing the number of dimensional variables into a smaller number of dimensionless parameters.
- Two of the commonly used methods are ~~the~~ (a) Raleigh's method, and (b) Buckingham Pi theorem method.

6.5.1 RALEIGH'S METHOD

- If A_1 is a dependent variable, and $A_2, A_3, \ldots A_n$ are independent variables in a phenomenon, then A_1 is expressed as

$$A_1 = k A_2^{a} A_3^{b} A_4^{c} \ldots\ldots A_n^{x}$$

when k is a dimensionless constant.

- The dimensions of each of the quantities $A_1, A_2, \ldots\ldots A_n$ are written and the sum of the exponents of M, L, T on both sides are equated.

- Care is needed in selecting the repeating variables:
  - (a) They must have amongst themselves, <u>all the basic dimensions</u> involved in the problem.
  - (b) <u>The dependent variable must NOT be chosen as a repeating variable</u>
  - (c) Usually a length parameter (such as a diameter), a typical velocity, and the fluid density, are a convenient set of repeating variables.