CHAPTER 4

Review of Vector analysis

Scalar: A quantity which has magnitude only. That is, it can be completely specified by a single number. Examples: mass, density, temperature.

A scalar is defined to be a tensor of order zero.

Vector: A quantity which is specified by both magnitude and direction. Examples: Force, Velocity, displacement.

A vector is defined as a tensor of order one.

Unit vector: A vector whose magnitude is unity. Thus, a unit vector $\vec{\nu}$ whose direction is parallel to a vector $\vec{A}$ can be written as $\vec{A} = \vec{\nu} A$, where $A = |\vec{A}|$

i.e. A is the magnitude of the vector $\vec{A}$.

Components of a vector

Let $\hat{i}, \hat{j}, \hat{k}$ represent unit vectors in the x, y, z directions, respectively, in a Cartesian coordinate system.

If $\vec{q}$ is chosen to be a vector in space, and u, v, w are the projections of $\vec{q}$ on the x, y, z axes, respectively, then

$$\vec{q} = \hat{i}u + \hat{j}v + \hat{k}w$$

where u, v, and w are the components of $\vec{q}$, and $\hat{i}u$, $\hat{j}v$, and $\hat{k}w$ are the corresponding component vectors.

The magnitude of $\vec{q}$ is

$$|\vec{q}| = \vec{q} = \sqrt{u^2+v^2+w^2}$$

And the direction cosines are

$$l = \cos\alpha = \frac{u}{|\vec{q}|}$$

$$m = \cos\beta = \frac{v}{|\vec{q}|}$$

$$n = \cos\gamma = \frac{w}{|\vec{q}|}$$

When $|\vec{q}| \neq 0$; otherwise we shall have the null vector.

Gradient : $\text{grad} \equiv \nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ ... (A.1)

Gradient of a scalar function $\phi(x,y,z)$ $\rightarrow$ $\nabla\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$ ... (A.2)

Where ϕ is a ~~grad~~ scalar and $\phi = \phi(x,y,z)$.

Material or Substantial derivative w.r.t. time

$$\frac{D}{Dt} = \underbrace{\frac{\partial}{\partial t}}_{\text{time variation at a fixed position (local derivative)}} + \underbrace{u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}}_{\text{convective derivative (change of a quantity due to movement of the particle)}} \quad \text{... (A.3)}$$

Acceleration of a fluid particle

$$\vec{a} = \frac{D}{Dt}\vec{V} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} \quad \text{... (A.4)}$$

$$= \frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V} \quad \text{... (A.5)}$$

In Cartesian coordinates, the acceleration components are

$$\left.\begin{aligned} a_x &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ a_y &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ a_z &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{aligned}\right\} \quad \text{... (A.6)}$$

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Divergence of a vector

If $\vec{V} = \hat{i}u + \hat{j}v + \hat{k}w$

then $$\operatorname{div}\vec{V} = \nabla\cdot\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad \text{--- (A-7)}$$

Curl (or rotation) of the vector function $\vec{V}$

$$\operatorname{curl}\vec{V} = \nabla\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \quad \text{---- (A-8)}$$

$$= \hat{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Vector identities

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi \quad \text{---- (A.9)}$$

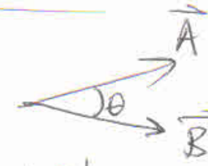
$$\nabla\cdot(\phi\vec{A}) = (\nabla\phi)\cdot\vec{A} + \phi(\nabla\cdot\vec{A}) \quad \text{---- (A.10)}$$

$$\nabla\times(\phi\vec{A}) = (\nabla\phi)\times\vec{A} + \phi(\nabla\times\vec{A}) \quad \text{--- (A.11)}$$

$$(\vec{A}\cdot\nabla)\vec{B} = A_x\frac{\partial\vec{B}}{\partial x} + A_y\frac{\partial\vec{B}}{\partial y} + A_z\frac{\partial\vec{B}}{\partial z} \quad \cdots\text{(A.12)}$$

Scalar multiplication of two vectors (dot product)

$$\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\cos(\vec{A},\vec{B}) = \underbrace{AB\cos\theta}_{\text{scalar}}$$



$\hat{i}\cdot\hat{i} = 1\cdot1\cdot\cos 0 = 1$ | $\hat{i}\cdot\hat{j} = 1\cdot1\cdot\cos 90^\circ = 0$

$\hat{j}\cdot\hat{j} = 1 = \hat{k}\cdot\hat{k}$ | $\hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

$|\vec{A}| = A$

$|\vec{B}| = B$

Divergence theorem (Gauss's theorem)

Let A denote a surface bounding a volume $\forall$, and $\hat{n}$ the unit vector normal to the surface, drawn outward. Then

$$\int_{\forall} \nabla\cdot\vec{q}\, d\forall = \int_{A} n\cdot\vec{q}\, dA$$

The following theorems are deductible from the Divergence theorem:

$$\int_{\forall} \nabla\phi\, d\forall = \int_{A} \hat{n}\phi\, d\vec{A}$$

$$\int_{\forall} \nabla\times\vec{q}\, d\forall = \int_{A} \hat{n}\times\vec{q}\, d\vec{A}$$

Stoke's Theorem

Let A be a surface bounded by a contour C, and $\hat{n}$ be the unit vector normal to the surface, in the sense which is positive when the right-hand rule wrt. to the direction of traversing the curve C is observed. Then

$$\int_{C} \vec{q}\cdot d\vec{r} = \int_{A} (\nabla\times\vec{q})\cdot\hat{n}\, dA = \int_{A} (\nabla\times\vec{q})\cdot d\vec{A}$$

CHAPTER 4. DIFFERENTIAL EQUATIONS OF FLUID FLOW

4.1 The differential equation of mass conservation

(4)

The principle of Conservation of Mass states that fluid mass cannot change. All the we apply this concept to a very small region. All the basic differential equations can be derived by considering either an elemental control volume or an elemental system. We choose an infinitesimal fixed control volume (dx, dy, dz) as in the figure below (Fig. 4.1).

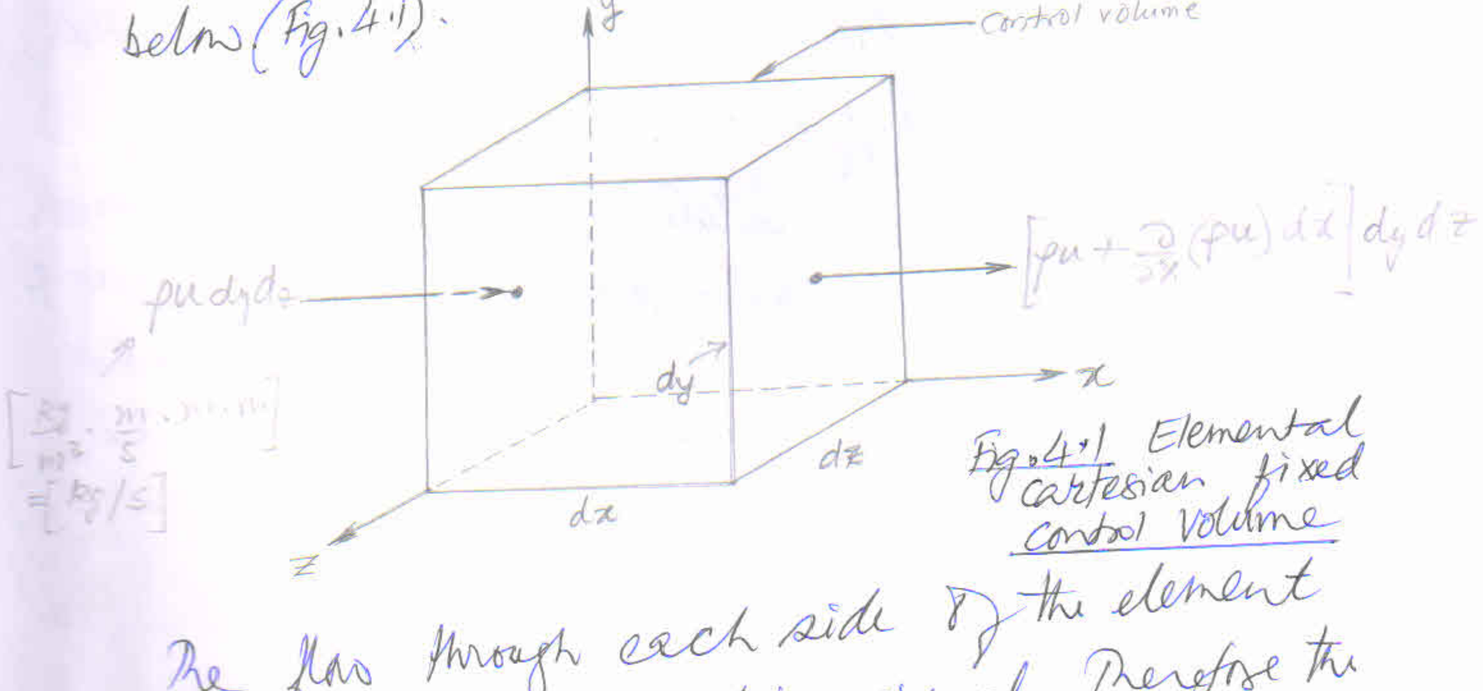


Fig. 4.1 Elemental cartesian fixed control volume

The flow through each side of the element is approximately one-dimensional. Therefore the appropriate mass conservation relation is

$$\int_{CV} \frac{\partial \rho}{\partial t}\, d\forall + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} = 0 \qquad (4.1)$$

volume

velocity

$\frac{\rho \cdot Vol}{time} = \frac{mass}{time}$

Mass flow rate $= \rho A V$

If the element considered is very small, the volume integral reduces to a differential term:

$$\int_{cv} \frac{\partial \rho}{\partial t}\, d\forall \cong \frac{\partial \rho}{\partial t}\, dx\, dy\, dz$$

Eqn. (4.1) then becomes

$$\frac{\partial \rho}{\partial t}\, dx\, dy\, dz + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} = 0 \quad (4.1a)$$

The mass flow occurs on all six faces: 3 inlets and 3 outlets. All fluid properties are considered to be uniformly varying functions of time and position, such as $\rho = \rho(x, y, z, t)$.

Thus, if T is the temperature on the left face of the element in Fig. 4.1, the right face will ~~be~~ have a temperature of $T + \left(\frac{\partial T}{\partial x}\right) dx$. For mass conservation, if (ρu) is known on the ~~f~~ left face, the value of this product on the right face is

$$\rho u + \frac{\partial}{\partial x}(\rho u) \cdot dx$$

Fig. 4.1 shows only the mass flow on the x or left and right faces. The flows on the y (bottom and top) faces have been omitted to avoid cluttering the drawing. We can list all these 6 flows as follows:

$\dot{m} = \rho A V$

Face	Inlet mass flow	Outlet mass flow
x	$\rho u \cdot dy \cdot dz$ (density·vel·area)	$\left\{\rho u + \frac{\partial}{\partial x}(\rho u)\,dx\right\} dy \cdot dz$
y	$\rho v\, dx\, dz$	$\left\{\rho v + \frac{\partial}{\partial y}(\rho v)\,dy\right\} dx \cdot dz$
z	$\rho w \cdot dx \cdot dy$	$\left\{\rho w + \frac{\partial}{\partial z}(\rho w)\,dz\right\} dx \cdot dy$

Introducing these terms into eqn. (4.1a), we get

$$\frac{\partial \rho}{\partial t}\,dx\,dy\,dz + \frac{\partial}{\partial x}(\rho u)\,dy\,dx\,dz + \frac{\partial}{\partial y}(\rho v)\,dx\,dy\,dz + \frac{\partial}{\partial z}(\rho w)\,dx\,dy\,dz = 0$$

The elemental volume $dx\,dy\,dz$ cancels out.

Unsteady
Viscous
compressible

$$\therefore \boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0} \quad (4.2)$$

This is the Conservation of mass for an infinitesimal control volume. This equation is also called the Equation of Continuity, where density and velocity are continuum functions. That is, the flow may be either steady or unsteady, viscous or frictionless, compressible or incompressible.

Using the vector gradient operator

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \quad \text{-----} (4.3)$$

the Equation of Continuity may be written as

· Unsteady
· Viscous
· compressible

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0} \quad \text{----} (4.4)$$

where $\nabla\cdot(\rho\vec{V}) = \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)$ ---- (4.5)

For steady compressible flow, eqn. (4.4) reduces to

$$\nabla\cdot(\rho\vec{V}) = 0 \quad \text{-------} (4.6a)$$

In cartesian coordinates

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \quad \text{----} (4.6b)$$

For steady incompressible flow, $\frac{\partial \rho}{\partial t} = 0$

∴ Eqn. (4.4) becomes

$$\nabla\cdot(\rho\vec{V}) = 0$$

i.e. $\nabla\cdot\vec{V} = 0$ ------ (4.7a)

In cartesian form this becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{----} (4.7b)$$

Equation of continuity - Conservation of mass
(An alternative formulation)

→ Mass can neither be created nor destroyed.

Fig 4.2

Referring to the figure above, moving surface A encloses a mass of fluid within volume ∀ lying entirely in the fluid.

If $\rho \Delta \forall$ is an element of mass, then the mathematical expression of the conservation of mass can be written as

$$\frac{D}{Dt}\left(\rho \Delta \forall\right) = 0 \quad \text{-------} (4.1')$$

$$\therefore \rho \frac{D}{Dt}(\Delta \forall) + \Delta \forall \frac{D\rho}{Dt} = 0$$

$$\text{or, } \frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{\Delta \forall}\frac{D}{Dt}\left(\Delta \forall\right) = 0 \quad \text{----} (4.2')$$

Eqn. (4.2') shows that a decrease in volume is accompanied by an increase in density and vice versa.

From (4.2) → $\dfrac{\Delta \forall}{\rho}\dfrac{D\rho}{Dt} + \dfrac{D}{Dt}(\Delta \forall) = 0$ --- (4.2a')

If we consider the total mass of fluid enclosed in the surface A, we get from (4.2a')

$$\lim_{\Delta \forall \to 0} \sum \left[\frac{1}{\rho} \frac{D\rho}{Dt} \Delta \forall + \frac{D}{Dt}\left(\Delta \forall\right) \right] = 0 \quad \cdots (4.3')$$

Now, $\vec{q} \cdot \nabla\rho = \left(\hat{i}u + \hat{j}v + \hat{k}w\right) \cdot \left(\hat{i}\frac{\partial\rho}{\partial x} + \hat{j}\frac{\partial\rho}{\partial y} + \hat{k}\frac{\partial\rho}{\partial z}\right)$

$$= \cancelto{1}{\hat{i}\cdot\hat{i}}\, u\frac{\partial\rho}{\partial x} + \cancelto{1}{\hat{j}\cdot\hat{j}}\, v\frac{\partial\rho}{\partial y} + \cancelto{1}{\hat{k}\cdot\hat{k}}\, w\frac{\partial\rho}{\partial z}$$

$$+ \left\{ \cancelto{0}{\hat{i}\cdot\hat{j},\ \hat{i}\cdot\hat{k},\ \hat{j}\cdot\hat{k}\ \text{terms}} \right\}$$

$$\therefore\ \vec{q} \cdot \nabla\rho = u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z} \quad \cdots\cdots (a)$$

Also, $\dfrac{D\rho}{Dt} = \dfrac{\partial\rho}{\partial t} + u\dfrac{\partial\rho}{\partial x} + v\dfrac{\partial\rho}{\partial y} + w\dfrac{\partial\rho}{\partial z}$ $\cdots\cdots$ (b) ← by definition of the material derivative

$= \dfrac{\partial\rho}{\partial t} + \vec{q}\cdot\nabla\rho$,

$\therefore$ The first term in (4.3') becomes, using (a)

$$\frac{1}{\rho}\frac{D\rho}{Dt}\Delta\forall = \frac{1}{\rho}\Delta\forall\left[\frac{\partial\rho}{\partial t} + \vec{q}\cdot\nabla p\right] \quad \cdots\cdots (c)$$

Then the first term of (4.3) can be written in the integral form, using (c)

$$\iiint \frac{1}{\rho}\left[\frac{\partial\rho}{\partial t} + \vec{q}\cdot\nabla p\right] d\forall \quad \cdots\cdots (4.4')$$

The second term in (4-3') represents the rate of change of volume.

The second term in (4.3') is

$$\lim_{\Delta \forall \to 0} \sum \frac{D}{Dt}(\Delta \forall)$$

It can be shown that

$$\nabla\cdot\vec{q} = \frac{1}{\delta\forall}\left[\frac{D}{Dt}(\delta\forall)\right]$$

$$\therefore \frac{D}{Dt}(\delta\forall) = (\nabla\cdot\vec{q})\,\delta\forall$$

$\therefore$ The second term in (4.3') becomes

$$\lim_{\Delta\forall\to 0}\sum\frac{D}{Dt}(\Delta\forall) = \iiint(\nabla\cdot\vec{q})\,d\forall \quad ---- (4.5')$$

Combining eqns. (4.3'), (4.4') and (4.5'), we find

$$\overbrace{\iiint\frac{1}{\rho}\left[\frac{\partial\rho}{\partial t} + \vec{q}\cdot\nabla\rho\right]d\forall}^{1^{st}\text{ term of }(4.3')}$$

$$+\overbrace{\iiint\nabla\cdot\vec{q}\,d\forall}^{2^{nd}\text{ term of }(4.3')} = 0$$

$$\text{or, } \iiint\frac{1}{\rho}\left[\frac{\partial\rho}{\partial t} + \vec{q}\cdot\nabla\rho + \rho(\nabla\cdot\vec{q})\right]d\forall = 0 \quad - (d)$$

$$\text{Now, } \vec{q}\cdot\nabla\rho = (\hat{i}u + \hat{j}v + \hat{k}w)\cdot\left(\hat{i}\frac{\partial\rho}{\partial x} + \hat{j}\frac{\partial\rho}{\partial y} + \hat{k}\frac{\partial\rho}{\partial z}\right)$$

$$\text{or, } \vec{q}\cdot\nabla\rho = u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z} \quad --- (e)$$

$$\text{And } \nabla\cdot\vec{q} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\hat{i}u + \hat{j}v + \hat{k}w)$$

$$\text{or, } \nabla\cdot\vec{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad --- (f)$$

and $\nabla\cdot(\rho\vec{q}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$

$$\cdot\left\{\hat{i}\,(\rho u) + \hat{j}\,(\rho v) + \hat{k}\,(\rho w)\right\}$$

$$= \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)$$

$$= \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial y} + v\frac{\partial \rho}{\partial y} + \rho\frac{\partial w}{\partial z} + w\frac{\partial \rho}{\partial z}$$

$$= \left(u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}\right) + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

or, $\nabla\cdot(\rho\vec{q}) = \overbrace{\vec{q}\cdot(\nabla\rho)}^{\text{from (e)}} + \rho\overbrace{(\nabla\cdot\vec{q})}^{\text{from (f)}}$ ------ (g)

Substituting back in (d)

$$\iiint \frac{1}{\rho}\left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{q})\right]dv = 0 \quad --- (4.6)$$

or, $\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{q}) = 0}$ ------ (4.7)

↳ Equation of continuity (General form)

In cartesian coordinates, eqn. (4.7) may be written as

Equation of continuity (in cartesian coordinate system)

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0} \quad \text{------} (4.9')$$

Eqn. (4.7) can also be written as

Equation of continuity (General form)

$$\boxed{\frac{D\rho}{Dt} + \rho\left(\nabla \cdot \vec{q}\right) = 0} \quad \text{--------} (4.8')$$

For incompressible flow $\frac{D\rho}{Dt} = 0$. Then eqn. (4.8') reduces to

$$\boxed{\nabla \cdot \vec{q} = 0} \quad \text{-----------} (4.10')$$

Equation of continuity for Incompressible flow

In Cartesian Coordinate system, eqn. (4.10') can be written as

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0} \quad \text{-----} (4.10a')$$

Eqn. of continuity (in cartesian coordinate system)

4.2 The Differential equation of Linear momentum

- This equation states that the vector sum of all external forces acting on a control volume (CV) in a fluid flow equals the time rate of change of linear momentum vector of the fluid mass in the CV.
- External forces are of two kinds, viz. boundary (surface) forces, and body forces.
- Boundary forces consist of
  - Pressure forces acting normal to a boundary (F_p)
  - Shear stresses acting tangential to a boundary (F_s)
- Body forces are those that depend upon the mass of the fluid in the CV, denoted by F_b. Example: Weight (gravity), magnetism, electric potential.
- The linear momentum equation in a general flow can be written for any direction x, as

$$\boxed{\sum F_x = F_{px} + F_{sx} + F_{bx} = \frac{\partial}{\partial t}\left(M_x\right)_{cv} + \dot{M}_{x\,out} - \dot{M}_{x\,in}} \quad \text{---(A)}$$

where $\dot{M}_x$ = momentum flux in the x-direction

$$= \underbrace{\rho Q V_x}_{\rho A V_x} = \dot{m} V_x \quad \left[Q = AV,\ \dot{m} = \rho AV = \rho Q\right) \quad \left(= \frac{kg}{m^3}\cdot m^2 \cdot \frac{m}{s} = \frac{kg}{s}\right)$$

$$\Rightarrow \underbrace{\rho A V}_{\dot{m}} V = \frac{kg}{m^3}\cdot m^2 \cdot \frac{m}{s}\cdot\frac{m}{s} = \frac{kg \cdot m}{s^2} = N$$

F_{px}, F_{sx} and F_{bx} represent the x-component of pressure force, shear force, and body force respectively, acting on the CV surface.

$\frac{\partial}{\partial t}(M_x)_{cv}$ = rate of change of x-momentum within the CV.

- Note that the momentum principle is derived from Newton's second law. The equation is applicable to a CV and are vector equations. (A CV is defined as a volume fixed in space).

We now use an elemental volume to derive Newton's Law for a moving fluid. (An alternate approach would be a force balance on an elemental moving particle.)

We use the same elemental control volume as in Fig. 4.1, for which the appropriate form of the linear momentum relation is

from (A) for x, y, z directions

$$\sum F = \frac{\partial}{\partial t}\left(\int_{cv} \vec{V} \rho \, d\forall\right) + \sum\left(\dot{m}_i \vec{V}_i\right)_{out} - \sum\left(\dot{m}_i V_i\right)_{in} \quad \text{---} (4.8)$$

The elemental volume is so small that the volume integral simply reduces to a derivative term:

$$\frac{\partial}{\partial t}\left(\int_{CV} \vec{V}\rho\, d\forall\right) \cong \frac{\partial}{\partial t}\left(\rho\vec{V}\right) dx\,dy\,dz \qquad ---(4.9)$$

The momentum fluxes occur on all six faces, three inlets and three outlets. Referring to Fig. 4.1, we can form a table of momentum fluxes by exact analogy with the equation for net mass flux:

$\frac{\text{momentum}}{\text{time}} = \frac{\text{mass.velocity}}{\text{time}}$

Faces	Inlet momentum flux	Outlet momentum flux
x	$\rho u \vec{V} dy\,dz$ ← ρAV	$\left\{\rho u\vec{V} + \frac{\partial}{\partial x}(\rho u\vec{V})dx\right\} dy\,dz$
	$\frac{\text{mass.velocity}}{\text{time}} = \frac{\text{mass.velocity}}{\text{time}}$	
y	$\rho v\vec{V}\ dx\,dz = \rho AV.V$	$\left\{\rho v\vec{V} + \frac{\partial}{\partial y}(\rho v\vec{V})dy\right\} dx\,dz$
z	$\rho w\vec{V}\ dx\,dy$	$\left\{\rho w\vec{V} + \frac{\partial}{\partial z}(\rho w\vec{V})dz\right\} dx\,dy$

Introduce these terms into eqn. (4.8). We get

$$\sum F = dx\,dy\,dz\left[\frac{\partial}{\partial t}(\rho\vec{V}) + \frac{\partial}{\partial x}(\rho u\vec{V}) + \frac{\partial}{\partial y}(\rho v\vec{V}) + \frac{\partial}{\partial z}(\rho w\vec{V})\right] \qquad ----(4.10)$$

This is a vector equation. A simplification occurs if we split the term in brackets as follows:

the square

$$\frac{\partial}{\partial t}(\rho\vec{V}) + \frac{\partial}{\partial x}(\rho u\vec{V}) + \frac{\partial}{\partial y}(\rho v\vec{V}) + \frac{\partial}{\partial z}(\rho w\vec{V})$$

Continuity eqn
$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0$

$$= \vec{V}\underbrace{\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V})\right]}_{=0 \text{ from Continuity eqn. (eqn. 4.4)}}$$

$$+ \rho\underbrace{\left(\frac{\partial\vec{V}}{\partial t} + u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}\right)}_{\frac{d\vec{V}}{dt}} \quad \text{---} (4.11)$$

The first terms within square brackets in (4.11) is the equation of Continuity (eqn. (4.4)), which therefore becomes zero.

The second term is seen to be the total acceleration of a particle that instantaneously occupies the CV (see eqn. (A.4), p. 46).

Then eqn. (4.10) reduces to

$$\boxed{\sum F = \rho\frac{d\vec{V}}{dt}\,dx\,dy\,dz} \quad \text{------} \quad (4.12)$$

Where $\frac{d\vec{V}}{dt}$ is the material or substantial derivative of $\vec{V}$, i.e. the material or substantial acceleration.

Body force

The body forces we will consider here is only due to gravity.

The gravity force on the differential mass $\rho\,dx\,dy\,dz$ within the CV is

$$dF_{grav} = \rho g\,dx\,dy\,dz \quad \text{----} \quad (4.13)$$

where $\vec{g}$ may in general have an arbitrary orientation w.r.t. the coordinate system.

Surface forces

The surface forces are due to the stresses on the sides of the control surface. These stresses are the sum of hydrostatic pressure plus viscous stresses τ_{ij} that arise from motion with velocity gradients:

pressure force

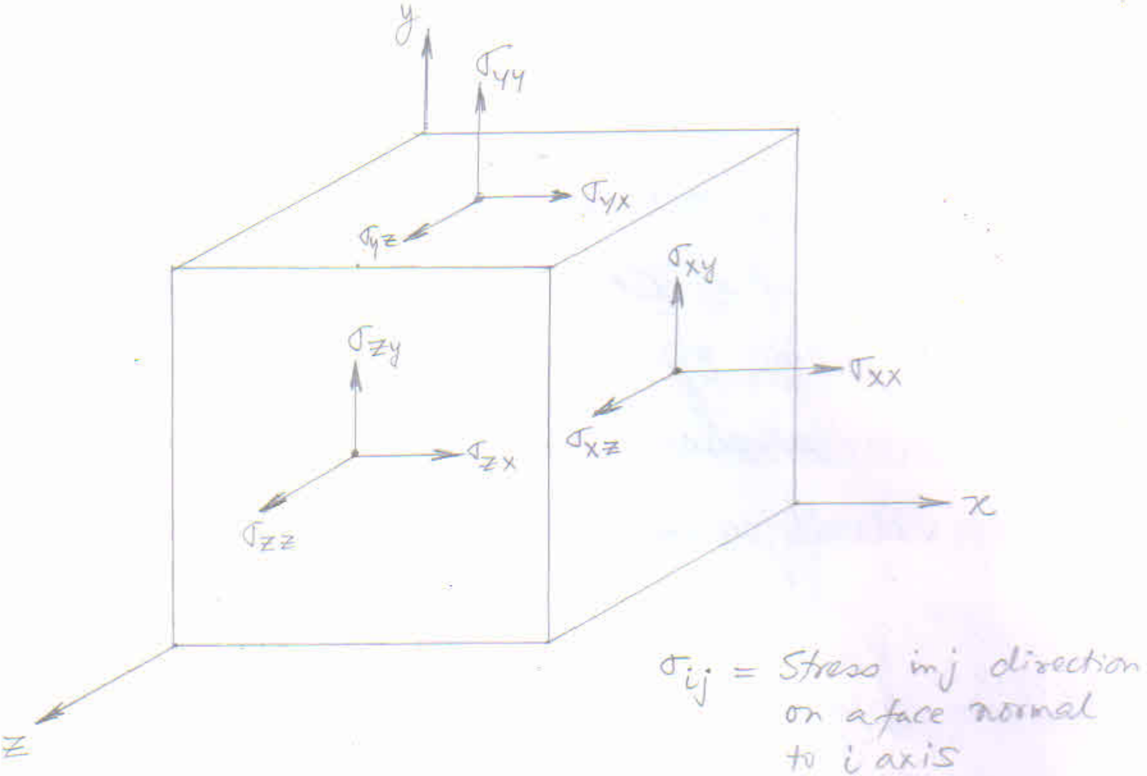
viscous stresses

Surface forces

$$\sigma_{ij} = \begin{vmatrix} -p+\tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p+\tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p+\tau_{zz} \end{vmatrix} \quad (4.14)$$

The subscript notation for stresses is given below in Fig. 4.3. Unlike velocity $\vec{V}$ which is a three-component vector, stresses σ_{ij} and τ_{ij} and strain rates ε_{ij} are nine-component tensors and require two subscripts to define each component.

It is not these stresses but their gradients, or differences, that cause a net force on the differential control surface. This can be seen in Fig. 4.4 which shows only the x-direction stresses to avoid cluttering the drawing.



<u>Fig 4.3</u> Notation for stresses

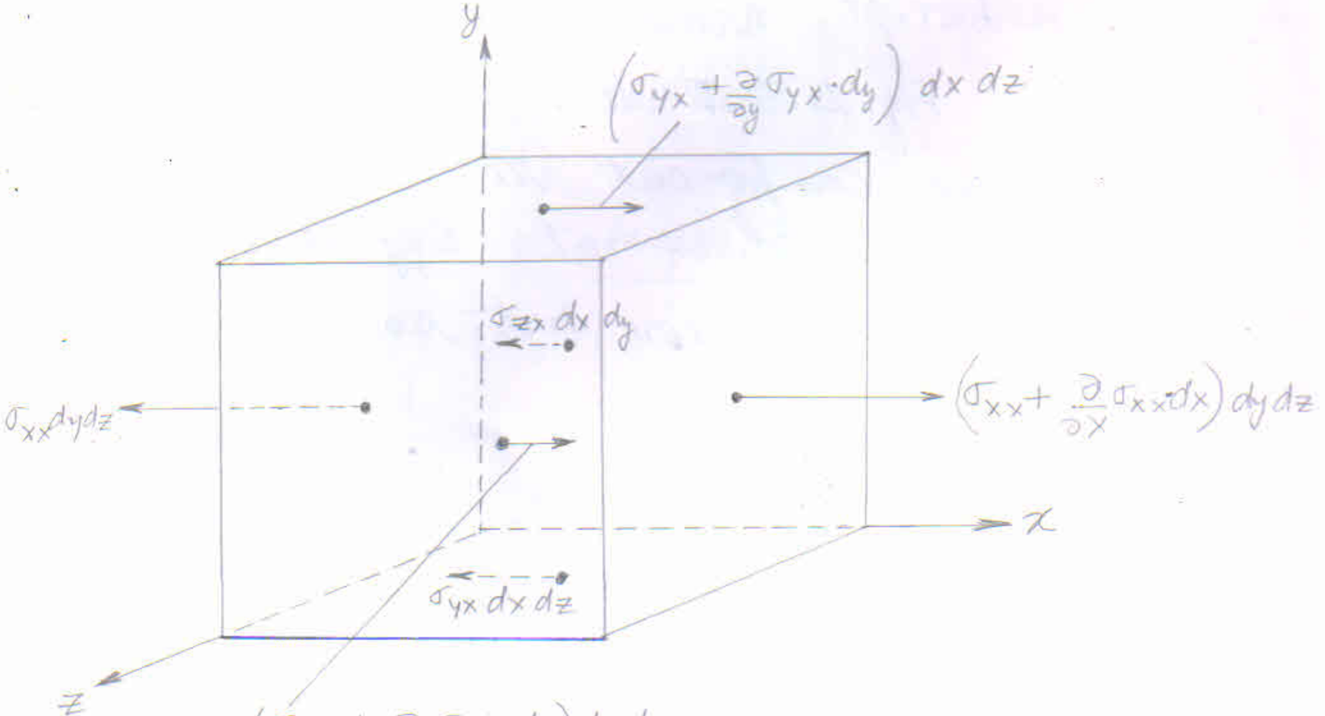


Fig. 4.4 Elemental cartesian fixed control volume showing surface forces in x-direction only

For example, the leftward force $\sigma_{xx}\,dy\,dz$ on the left face is balanced by the rightward force $\sigma_{xx}\,dy\,dz$ on the right face, leaving only the net rightward force $\left(\frac{\partial}{\partial x}\sigma_{xx}\right)dx\,dy\,dz$

on the right face.

The same thing happens in the other four faces, so that the net surface force in the x-direction is given by

$$dF_{x,surf} = \left[\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx})\right] dx\,dy\,dz \quad \text{---} \; (4.15)$$

We see that this force is proportional to the elemental volume. Note that the stress terms are taken from the top row of the array in eqn. (4.14). Splitting this row into pressure plus viscous stresses, we can rewrite eqn. (4.15) as

Surface forces

$$\frac{dF_x}{d\forall} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx}) \quad \text{----} \; (4.16)$$

where $d\forall = dx\,dy\,dz$.

In exactly similar manner, we can derive the y & z forces per unit volume on the control surface:

$$\frac{dF_y}{d\forall} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\tau_{xy}\right) + \frac{\partial}{\partial y}\left(\tau_{yy}\right) + \frac{\partial}{\partial z}\left(\tau_{zy}\right)$$

$$\frac{dF_z}{d\forall} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left(\tau_{xz}\right) + \frac{\partial}{\partial y}\left(\tau_{yz}\right) + \frac{\partial}{\partial z}\left(\tau_{zz}\right)$$

--- (4.17)

We now multiply eqns. (4.16) and (4.17) by the unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$, respectively, and add to obtain an expression for the net vector surface force:

$$\left(\frac{d\vec{F}}{d\forall}\right)_{surf} = -\nabla p + \left(\frac{dF}{d\forall}\right)_{viscous} \quad \text{--- (4.18)}$$

where the viscous force has a total of nine terms:

$$\left(\frac{d\vec{F}}{d\forall}\right)_{viscous} = \hat{i}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + \hat{j}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + \hat{k}\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right)$$

---- (4.19)

Since each term in parantheses in eqn. (4.19) represents the divergence of a stress component vector acting on the x, y, and z faces respectively, eqn. (4.19) is sometimes expressed in divergence form:

$$\left(\frac{d\vec{F}}{d\forall}\right)_{viscous} = \nabla\cdot\vec{\vec{\tau}}_{ij} \quad \cdots\cdots (4.20)$$

where
$$\tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix} \quad \cdots\cdots (4.21)$$

τ_{ij} is the viscous stress tensor acting on the element.

The surface force is thus the sum of the pressure gradient vector and the divergence of the viscous stress tensor.

Substituting into eqn. (4.12) and utilizing eqn. (4.13), we have the basic differential momentum equation for an infinitesimal element:

$$\boxed{\rho\vec{g} - \vec{\nabla}p + \vec{\nabla}\cdot\vec{\vec{\tau}}_{ij} = \rho\frac{d\vec{V}}{dt}} \quad \cdots\cdots (4.22)$$

where
$$\frac{d\vec{V}}{dt} = \frac{\partial\vec{V}}{\partial t} + u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z} \quad \cdots\cdots (4.23)$$

Eqn. (4.22) ~~(4.23)~~ can be expressed physically as follows:

$$\boxed{\begin{array}{l}\text{Gravity force per unit volume}\\ \quad + \text{pressure force per unit volume}\\ \quad\quad + \text{viscous force per unit volume}\\ = \text{density} \times \text{acceleration}\end{array}} \quad (4.24)$$

Eqn. (4.22) is so brief & compact that its inherent complexity is invisible. It is a <u>vector</u> equation, each of whose component equations contains <u>nine</u> terms.

The component equations can be written out in full as follows:

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) \quad (4.25)$$

$$\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Differential momentum equation valid for any fluid in any general motion

Eqn. (4.25) is the differential momentum equation in its full glory, and it is valid for <u>any fluid in any general motion, particular fluids being characterized by particular viscous stress terms.</u>

Note that the last three "convective" terms on the right hand side of each component equation in (4.25) are non-linear, which complicates the general mathematical analysis.

<u>Inviscid flow : Euler's equation</u>

Eqn. (4.25) is not ready to use until we write the viscous ~~terms in~~ stresses in terms of velocity components.

The simplest assumption is frictionless flow, i.e. $\tau_{ij} = 0$, for which eqn. (4.22) reduces to

$$\rho\vec{g} - \vec{\nabla}p = \rho\frac{d\vec{V}}{dt} \quad \text{------} \quad (4.26)$$

This is <u>Euler's eqn. for inviscid flow.</u>

Euler's eqn. can be integrated along a streamline to yield <u>the frictionless Bernoulli equation</u>, which is

[illegible]

Bernoulli's eqn. for unsteady, frictionless flow along a streamline

$$\boxed{\frac{\partial V}{\partial t}ds + \frac{dp}{\rho} + V\,dV + g\,dz = 0} \quad \text{---} (4.27)$$

where $\{S$ = distance along streamline, V = velocity

This is Bernoulli's equation for unsteady frictionless flow along a streamline. This can be integrated between any two points 1 and 2 on the streamline:

$$\boxed{\int_1^2 \frac{\partial V}{\partial t}ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}\left(V_2^2 - V_1^2\right) + g(z_2 - z_1) = 0} \quad \text{---} (4.28)$$

For steady $\left(\frac{\partial V}{\partial t} = 0\right)$, incompressible (constant density) flow, eqn. (4.28) becomes

$$\frac{p_2 - p_1}{\rho} + \frac{1}{2}\left(V_2^2 - V_1^2\right) + g(z_2 - z_1) = 0.$$

Bernoulli eqn. for steady, frictionless, incompressible flow along a streamline

or, $$\boxed{\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + g z_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + g z_2 = \text{Const.}} \quad \text{---} (4.29)$$

This is the <u>Bernoulli equation</u> for <u>steady</u>, <u>frictionless</u> <u>incompressible</u> flow along a <u>streamline</u>.

The Bernoulli eqn. (4.29) is a classic momentum result, Newton's law for a frictionless, incompressible fluid. It may also be interpreted, however, as an idealized energy relation. The changes from 1 to 2 in eqn. (4.29) represent reversible pressure work, kinetic energy change, and potential energy change. The fact that the total remains the same means that there is no energy exchange due to viscous dissipation, heat transfer, or shaft work. These effects can be added by making a control volume analysis of the first Law of Thermodynamics.

Newtonian fluid: Navier–Stokes equations

For a Newtonian fluid, the viscous stresses are proportional to the element strain rates and the coefficient of viscosity. For incompressible flow, the generalization of the relation

$$\tau = \mu \frac{du}{dy}$$

to three dimensional ~~flow~~ viscous flow is

For incompressible 3D viscous flow

$$\left.\begin{aligned}
&\tau_{xx} = 2\mu \frac{\partial u}{\partial x} \; ; \quad \tau_{yy} = 2\mu \frac{\partial v}{\partial y} \; ; \quad \tau_{zz} = 2\mu \frac{\partial w}{\partial z} \\
&\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right); \quad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\
&\qquad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)
\end{aligned}\right\} \quad (4.30)$$

where μ is the viscosity coefficient.

Substitution of the above in eqn. (4.25) gives the differential momentum equation for a Newtonian fluid with constant density & viscosity:

incompressible N-S eqns. with constant ρ & μ

$$\left.\begin{aligned}\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) &= \rho\frac{du}{dt}\\ \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) &= \rho\frac{dv}{dt}\\ \rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) &= \rho\frac{dw}{dt}\end{aligned}\right\} \quad (4.31)$$

The above are the incompressible Navier-Stokes equations. They are second-order nonlinear partial differential equations (pde's) and are formidable, but solutions have been found for a variety of interesting viscous flow problems.

Eqn. (4.31) have 4 unknowns : p, u, v, w. They should be combined with the incompressible continuity equation (4.7b p.52) to form four equations in these four unknowns.

Even though the Navier-Stokes equations have only a limited number of known analytic solutions, they are amenable to fine-gridded computer modeling. It is now possible to achieve approximate, but realistic, CFD results for a wide variety of complex two- and three-dimensional viscous flows.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{(4.7b p.52)}$$

.3 The differential equation of energy.

The appropriate integral relation for the fixed control volume of Fig. 4.1 (P.49) is

$$\dot{Q} - \dot{W}_s - \dot{W}_v = \frac{\partial}{\partial t}\left(\int_{cv} e\rho \, d\mathcal{V}\right) + \int_{cs}\left(e+\frac{p}{\rho}\right)\rho\left(\vec{V}\cdot\hat{n}\right)dA \quad (4.32)$$

where $\dot{W}_s$ = shaft work = 0, because there can be no infinitesimal shaft protruding into the control volume.

By analogy with eqn. (4.10) p.60, the right-hand side for the tiny element becomes

$$\dot{Q} - \dot{W}_v = \left[\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x}(\rho u \zeta) + \frac{\partial}{\partial y}(\rho v \zeta) + \frac{\partial}{\partial z}(\rho w \zeta)\right] dx\,dy\,dz$$

where $\zeta = e + \frac{p}{\rho}$.

When we use the continuity eqn. by analogy with eqn. (4.11) p.61, the above eqn. becomes

$$\boxed{\dot{Q} - \dot{W}_v = \left(\rho\frac{\partial e}{\partial t} + \vec{V}\cdot\vec{\nabla}p + p\vec{\nabla}\cdot\vec{V}\right) dx\,dy\,dz} \quad (4.33)$$

This is the differential equation of energy.